Social Structure and Redistributive Politics

Perry Carter Joseph J. Ruggiero

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Abstract

We propose a model of electoral competition where candidates offer excludable transfers to voters connected on a social network. We derive concentration results for weighted graphs to compare large societies, with social structure represented by asymptotic properties. We find that the diversion of public resources towards private provision generically favors the majority group and is least prevalent when groups are equipopulous and segregated. If candidates have heterogeneous information about the electorate, information can overcome the tyranny of the majority when groups are segregated or, given weak peer effects or an overall poor quality of information, candidate office motivations are low.

JEL: D72, D85, O17, Z13

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Carter: Princeton University, Fisher Hall, Princeton, NJ 08544 (email: pjcarter@princeton.edu); **Ruggiero:** Princeton University, Fisher Hall, Princeton, NJ 08544 (email: jruggiero@princeton.edu).

Politicians compete by making campaign promises, which often include targeted attempts to induce individuals to vote. Transfers encourage voters but do not fully govern their decisions. In addition to policy platforms and public good provision, a voter's choice is influenced by their social connections, making a candidate's ability to buy any given voter dependent on how anchored that voter is by the opinions of their peers. However, this implies that candidates can also indirectly sway voters by using private transfers to sway their friends. As a consequence, the structure of society regulates the form and efficacy of redistributive politics. Then, which societies are more or less conducive to redistributive electoral strategies?

We study a model of a large election in which candidates compete to influence policy-motivated voters with targeted and excludable transfers, which take the form of campaign promises. The model borrows from the framework in Battaglini and Patacchini (2018), who study campaign contributions and the role of pivotal voting in small legislative elections. We additionally incorporate a trade-off between redistribution and a public good; however, our primary contribution is in using the model to systematically compare redistributive strategies across different large societies through the use of random graphs.

At the core of the model is network dependence: voters not only value direct transfers, but also those extended to their peers. The effectiveness of targeted redistribution depends on the ability of candidates to exploit differential benefits of swaying some voters relative to the losses this induces from all others due to diversion of public resources. At the same time, candidates must also avoid triggering negative spillovers when influential voters are dissatisfied with allocations. The unique equilibrium transfer to a voter is proportional to a measure of network centrality, which captures the incentive to target voters who influence others. Similarly to previous work on finite population noncooperative games (Ballester, Calvó-Armengol and Zenou, 2006; Battaglini, Sciabolazza and Patacchini, 2020; Chen, Zenou and Zhou, 2022), we derive a result in which a public good trade-off causes transfers to be proportional to each individual's share of the total centrality on the network.

Most importantly, the model is used to explore the effects of social structure in large general elections. Because centrality is sensitive to small changes on the network, recovering a comprehensive account of how equilibrium behavior responds to changes in society's underlying structural features would require accounting for all possible realizations of society given those underlying features, which is neither theoretically tractable nor empirically possible. To overcome this problem, we incorporate stochastic block models, where deep parameters that govern network formation correspond to the structural features of a society. By extending techniques for random graph analysis (Chung and Radcliffe, 2011; Dasaratha, 2020; Mostagir and Siderius, 2021), we derive results that allow us to substitute complex weighted graphs with their expected counterparts and derive closed-form expressions for expected equilibrium strategies. This creates a direct mapping from the fixed characteristics that form social networks to expected equilibrium behavior.

Focusing on candidate strategies on the expected network allows us to draw sharp conclusions about the effects of social structure that hold asymptotically in large societies. Specifically, we show that expected equilibrium strategies are necessarily arbitrarily close to the equilibrium strategies of any realization, making this analysis possible. Obtaining expressions that depend on the social structure, and not on the arbitrary structure of a specific realization, then permits the systematic comparison of different societies.

Our baseline results suggest that diversion of public resources will be most prevalent when groups are integrated and there is a significant size disparity between them. The similarity in group size—or fractionalization (Alesina et al., 2003)—matters not because of differences in policy preferences as in Easterly and Levine (1997) and Alesina, Baqir and Easterly (1999), but because it affects the proportion of potential ties that are within or between groups and hence the level of social pressure that can be achieved. Segregation, on the other hand, typically decreases the value of transfers as it reduces the candidate's expected return—as a voter becomes less connected to individuals outside their group, their ability to influence members of the other group is attenuated.

Finally, we also consider the case where candidates enjoy more accurate information about the members of one group, creating an ex-ante incentive to target them. We are primarily interested in understanding when the effect of information on candidate strategies is in tension with that of the network from our baseline. While a general feature of the baseline model is that members of the majority group attract greater investment due to their greater potential to influence others, we find that better information on minorities can make members of their group more attractive despite the network effect, overcoming the "tyranny of the majority." This information effect will dominate the pure network effect when segregation is low, as the tendency of voters to sort into isolated social groups is a primary source of inefficiency from the candidate's perspective. Moreover, the information effect may prevail when candidate office-motivations are low and the expected marginal gain from network spillovers is sufficiently low. This is because a reduced incentive to redistribute at the expense of a public good causes candidates to prefer focusing resources on more predictable voters.

Related Literature.—Our paper is most closely related to work in electoral competition and redistributive politics. Empirical evidence has shown that politicians strategically target individual voters for electoral gain (Enikolopov, 2014; Alatas et al., 2019). To understand this phenomenon, theoretical work has explored a number of mechanisms governing redistributive strategies: the incentives for candidates to create inequalities under different electoral systems (Myerson, 1993), the ability of excludable transfers to induce inefficient supermajority coalitions (Groseclose and Snyder, 1996; Banks, 2000), the difficulty of overcoming private incentives by providing public goods (Lizzeri and Persico, 2001), the role of varying commitment structures (e.g., campaign promises or up-front voting) and institutions in mitigating inefficiencies in redistribution (Dekel, Jackson and Wolinsky, 2008), and the institutional factors driving the strategies of clientelist machines (Gans-Morse, Mazzuca and Nichter, 2014).

In these previous models, variation in electoral redistributive strategies has been driven by either individual factors or institutional environments. Explanations from empirical studies, however, have regularly emphasized the role of social connections. For example, recent studies have provided evidence that politicians are responsive to micro-level social network structure when deciding how to allocate targeted transfers, such as political connections (Caeyers and Dercon, 2012; Fafchamps and Labonne, 2017) or social influence (Cruz, 2019; Cruz, Labonne and Querubin, 2020).

We incorporate these observations into a model of individually targeted redistribution on a known social network. In this regard, we build on the innovations of recent work in the political economy of networks that exploit local linearities in agent strategies to generate tractable results. This includes studies on the effects of network topology on information spread and learning (Golub and Jackson, 2012; Canen, Schwartz and Song, 2020), pricing under oligopolistic competition (Chen, Zenou and Zhou, 2022), public goods provision (Elliott and Golub, 2019), and legislative activities (Battaglini, Sciabolazza and Patacchini, 2020; Canen, Jackson and Trebbi, 2023).

With regard to the game played on a realized graph, the closest model to ours is offered by Battaglini and Patacchini (2018), who study small legislative elections. Specifically, we adopt the approach in which voters are influenced by an average of their peers' probabilities of taking an action (Calvó-Armengol, Patacchini and Zenou, 2009; Lee et al., 2021). We depart from previous work by considering an expected rather than a fixed graph, permitting direct analysis of the impact of average features of society on equilibrium strategies.

To do so, we employ stochastic block models, which have been widely used to understand social phenomena and have been shown to perform well in approximating real social networks (Ghasemian, Hosseinmardi and Clauset, 2019; Vaca-Ramírez and Peixoto, 2022). Two noteworthy examples in economics include Sadler (2023), which incorporates stochastic block models to extend the fixed-graph game of Yildiz et al. (2013), and Golub and Jackson (2012), which uses a variant to study the impact of network homophily on learning. While we focus on a model of redistributive politics in the present analysis, social network structure is important in many areas of social science, and our overall approach can easily be extended to other applications such as consumer choice. For instance, sellers may seek to offer targeted prices to consumers who are influenced by their social connections' consumption behavior (Wang and Wang, 2017). Our framework would then allow us to study the features of societies that lead to higher overall prices and to more or less price discrimination.

I. Model

We begin by outlining the model for a fixed graph, which resembles the approach in Battaglini and Patacchini (2018). In the sections thereafter, we turn to our primary task of exploring the role of social structure through the use of random graphs.

There are n voters faced with a choice between two candidates. All voters are located on a network $\mathcal{G}$, which is assumed to be connected. We use the terms network and graph interchangeably throughout the paper to refer to a directed weighted graph, which is an ordered triple $(\mathcal{V}, \mathcal{E}, w)$ where $\mathcal{V}$ is a set of n vertices (representing voters) and $\mathcal{E} \subseteq \{\{i,j\} : i,j \in \mathcal{V} \wedge i \neq j\}$ is a set of directed edges with corresponding weight function $w : \mathcal{E} \to \mathbb{R}_+$. Each candidate $k = 1,2$ is associated with a policy $y_k = k$, where the policy space is a subset of $\mathbb{R}$, and each voter $i \in \mathcal{V}$ is endowed with a group membership $\ell_i = 1,2$, which corresponds to an ideal policy $x_i = \ell_i$. Substantively, groups may be interpreted as corresponding to any grouping that is both socially and politically meaningful, such as political parties and ethnic or religious groups.

To gain vote share, candidate k can extend n private transfers, $b_{ik} \geq 0$. These payments, however, come at the expense of a public good, which the candidate also values. A candidate k's problem is to choose $\boldsymbol{b}_k \in \mathbb{R}^n_+$ that solves

$$\text{(P)} \qquad \max_{\boldsymbol{b}_k} \ \alpha_k \sum_{i \in \mathcal{V}} \phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k}) - \boldsymbol{b}_k \cdot \mathbf{1}^\top \quad \text{subject to } \ b_{ik} \geq 0 \text{ for all } i$$

where $\boldsymbol{b}_{-k} \in \mathbb{R}^n_+$ are the transfers offered by the opposing candidate, $\phi_{ik}(\cdot) \in (0,1)$ is the probability voter i votes for candidate k net of transfers, and $\alpha_k > 0$ represents the value placed on one vote by candidate k. We thus normalize the value placed on a unit of public good by the candidate to 1 so that α_k can be interpreted as the candidate's relative degree of office motivation. In particular, a small α_k implies that candidate k highly values the public good, while larger α_k implies greater office motivation.

The expected payoff voter i receives from candidate k is given by

$$\text{(1)} \qquad U_i(k) = -(x_i - y_k)^2 + u(b_{ik}) + \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}\phi_{jk}(\boldsymbol{b}) - \gamma \sum_{m \in \mathcal{V}} b_{mk} + \varepsilon_{ik}$$

where $u(\cdot)$ is voter utility over transfers, which we assume satisfies $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $u'''(\cdot) \geq 0$, as well as $\lim_{b \to 0} u'(b) = \infty$ and $\lim_{b \to \infty} u'(b) = 0$. In subsequent sections, we will assume logarithmic utility for comparative statics.

Voters support the candidate that offers them a higher total utility. There is no obligation to vote for a candidate who offered them a payment, but payment is received after the election and are thus campaign promises in the sense of Dekel, Jackson and Wolinsky

(2008). First, all voters care about policy and a public good. Policy preferences follow a standard quadratic loss function and value a unit of public good at $\gamma \geq 0$, so that they incur a loss of γ for every unit of transfers offered by a candidate to any voter.

Second, voters have private information in the form of a valence shock for each candidate, $\varepsilon_{ik} \in \mathbb{R}$. Without loss of generality, we can normalize $\varepsilon_{i2} = 0$ and define $\varepsilon_i := \varepsilon_{i1}$, which we assume is an independent, uniformly distributed mean-zero random variable with support on $\left[\frac{1}{-2\theta}, \frac{1}{2\theta}\right]$. We interpret θ as the quality of candidate information about the utility of transfers to voters, with smaller θ indicating less informed candidates.

Third, voters also like to vote in alignment with their social connections. $\phi_{ik}(\cdot)$ denotes the probability voter i votes for candidate k given all transfers, but before the realization of the valence shock ε_i. Then, each voter i places weight $w_{ij} > 0$ on voter j's probability of voting for candidate k if there is a directed edge from voter i to j, and 0 otherwise. In the graph $\mathcal{G}$, the set of a voter i's peers is denoted by $\mathcal{T}_i(\mathcal{G}) \subseteq \mathcal{V}$.

For the solution to (P) to be well-defined, we make two assumptions on θ.

Assumption 1. $\max_{\boldsymbol{b}_k} \theta \left(u(b_{ik}) + 1 - \gamma \sum_i b_{ik}\right) < \frac{1}{2}$ *for all* i, k*.*

With our conditions on $u(\cdot)$, this ensures that all voters have a well-defined interior probability of voting for either candidate for any transfer profile and is equivalent to a condition that θ is sufficiently small.

Before stating the second assumption, we must first define the following object.

Definition 1. *Consider a realized graph* $\mathcal{G}$ *and a corresponding adjacency matrix* $\boldsymbol{A}$ *such that for all* $i, j \in \mathcal{V}$*,* $A_{ij} = 1$ *if* $j \in \mathcal{T}_i(\mathcal{G})$ *and* $A_{ij} = 0$ *otherwise. Then, the induced adjacency matrix* $\tilde{\boldsymbol{A}}$ *is given by, for all* $i, j \in \mathcal{V}$*,* $\tilde{A}_{ij} = w_{ij} A_{ij}$*.*

The induced adjacency matrix accounts for the fact that a voter may be more influenced by one peer than another. Letting $\boldsymbol{I}$ denote the identity matrix, we also assume the following.

Assumption 2. $\boldsymbol{I} - 2\theta\tilde{\boldsymbol{A}}$ *is invertible.*

Assumption 2 is equivalent to assuming that θ is smaller than $\frac{1}{2\lambda_1}$, where λ_1 is the largest eigenvalue of the matrix $\tilde{\boldsymbol{A}}$. The largest eigenvalue is bounded by $n \max_{ij} w_{ij}$, making $\theta \leq \frac{1}{2n \max_{ij} w_{ij}}$ sufficient. Since transfers are in units of utility over policy, this does not imply transfers must be small.

Timing.—The game proceeds as follows.

(i) Nature randomly chooses a private utility shock for each voter, $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$.

(ii) For all voters $i \in \mathcal{V}$, each candidate $k = 1, 2$ offers a payment $b_{ik} \geq 0$.

(iii) Each voter $i \in \mathcal{V}$ casts a ballot for candidate 1 or 2, $v_i = 1, 2$.

(iv) The winning candidate enacts their promised transfer program and voters' utility is realized.

A. Equilibrium

A voter will cast a ballot for candidate 1 if and only if $U_i(1) \geq U_i(2)$. Here, candidates will not be able to perfectly anticipate voting behavior due to their imperfect information over voter preferences. Using equation (1), we can rewrite this as a condition on the size of the valence shock,

$$(-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}(2\phi_j - 1) + \gamma \sum_{m \in \mathcal{V}} (b_{m2} - b_{m1}) \geq \varepsilon_i,$$

where we have denoted $\phi_i := \phi_{i1}(\boldsymbol{b}) = 1 - \phi_{i2}(\boldsymbol{b})$ the probability a voter i votes for candidate 1. Noting that $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$ implies $\Pr(\varepsilon_i \leq \varepsilon) = \frac{1}{2} + \theta\varepsilon$, we can correspondingly write each voter's probability for voting for candidate 1 as, for all $i \in \mathcal{V}$,

$$\phi_i = \frac{1}{2} + \theta\left((-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}(2\phi_j - 1) + \gamma \sum_{m \in \mathcal{V}} (b_{m2} - b_{m1})\right). \tag{2}$$

Here, $\boldsymbol{\phi}$ gives the unique vector of vote probabilities given payment profiles. While each voter's utility is subject only to their neighbor's vote probabilities, this system of equations necessarily implies that a single voter's probability of supporting candidate 1 is a function of all other voter's probability of supporting 1. This occurs because, for example, a voter i's probability ϕ_i is affected by i's neighbor j's probability ϕ_j, which in turn is affected by j's neighbor m's probability ϕ_m. Because we rule out disconnected components, ϕ_i will both affect and be affected by all other voting probabilities throughout the entire network.

In equilibrium, each candidate chooses a vector of transfers that maximizes their utility, taking the other candidate's strategy as given. This gives rise to the n first-order conditions,

$$\sum_{j=1}^{n} \frac{\partial \phi_j}{\partial b_{ik}} = \frac{1 - \lambda^*_{ik}}{\alpha_k}$$

where λ^*_{ik} is the Lagrange multiplier associated with the nonnegativity constraint for an i, k pair (not to be confused with eigenvalues λ). Differentiating equation (2), we have

$$\frac{\partial \phi_i}{\partial b_{h1}} = \theta\left(u'(b_{h1})\mathbb{1}(i = h) + 2\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}\phi'_j - \gamma\right).$$

Thus, the candidates' problem can be rewritten as

$$(\boldsymbol{J}\left[\boldsymbol{u}\right] - \boldsymbol{\Gamma})^{\top} \cdot \left(\boldsymbol{I} - 2\theta\tilde{\boldsymbol{A}}\right)^{-1} \cdot \boldsymbol{1} = \frac{(1 - \boldsymbol{\lambda}^*)}{\alpha_k\theta} \tag{3}$$

where $\boldsymbol{J}[\cdot]$ is a diagonal matrix with $u'(b_i)$ as the nonzero entries, $\boldsymbol{\Gamma}$ is an $n \times n$ square matrix with γ as every element, $\mathbf{1}$ denotes an n-vector of ones, and $\boldsymbol{\lambda}^*$ is an n-vector of Lagrange multipliers.

From the candidates' problem in equation (3), we can recover candidate k's equilibrium transfer to voter i,

$$b_{ik} = \left[u'\right]^{-1} \left(\frac{1}{c_i(\boldsymbol{w}, \theta; \mathcal{G})} \left[\gamma C(\boldsymbol{w}, \theta; \mathcal{G}) + \frac{1}{\alpha_k \theta} \right] \right), \tag{4}$$

where $c_i(\cdot)$ is the ith element of $\boldsymbol{c} = (\boldsymbol{I} - 2\theta\tilde{\boldsymbol{A}})^{-1} \cdot \mathbf{1}$ and $C(\cdot) := \sum_{i \in \mathcal{V}} c_i(\cdot)$. Here, $\boldsymbol{c}$ is equivalent to Katz-Bonacich centrality on the weighted directed network corresponding to $\tilde{\boldsymbol{A}}$ with attenuation parameter 2θ. The value of a voter is thus proportional to their share of total centrality on the graph, with effects on more distant connections attenuated by the predictability of each voter's behavior, θ.

This behavior is intuitive given the basic strategic environment: the socially optimal transfers to voters would correspond to a transfer scheme such that the marginal value is equated with γ, weighted by the total centrality on the network, which can be taken as a measure of the additional positive spillover candidates gain from providing a public good. Network spillovers also incentivize candidates to provide additional transfers beyond this level, however. The network induces a trade-off between the positive effect associated with providing a transfer to i, which are captured by c_i and the concomitant negative effect this induces via *every other voter* being deprived of the public good, captured by C.

The relative value of these spillovers is moderated by α_k—that is, more office-motivated candidates place higher value on the additional increase to expected vote share afforded by targeting high-centrality voters—and by θ, which determines the certainty of a realized increase in vote share. Finally, it is not necessarily true that an increase in the number of voters results in a decrease in transfers, since it may be possible to add another voter in such a way that centrality increases for some i due to the creation of new paths.

Disconnected Model.—In the case $\mathcal{G}_0$ where the network is completely disconnected (or, equivalently, where it is completely connected with weights such that all voters have an equal centrality of 1), this reduces to simply

$$b_{ik} = \left[u'\right]^{-1} \left(\frac{\gamma \alpha_k \theta n + 1}{\alpha_k \theta} \right) \tag{5}$$

From this, it is apparent that the candidate's office motivation α_k has two competing effects on equilibrium transfers: candidates have an incentive to provide more transfers to individuals, but also to offer less to everyone due to the penalty voters impose on reduction in public good provision. Importantly, the former effect will always prevail, so that office-motivated candidates will always allocate more resources to private transfers even as the value of public goods approaches infinity. When there is heterogeneous information about the preferences of individual voters, however, the balance of these two channels is more subtle, a possibility we take up in Section IV.

It is also of note that, if both candidates have identical degrees of office motivation ($\alpha_1 = \alpha_2 = \alpha$), then they will also choose the same transfer profile in equilibrium. This can be thought of as an analogue to the Median Voter Theorem for spatial models of competition (Downs, 1957); it is optimal for both candidates to respond by extending offers to the voters that offer the most value, here determined entirely by their network positions. A further implication is then that targeted distribution only influences aggregate electoral outcomes if candidates diverge in their motivations. When candidates are perfectly symmetric, their offers exactly offset one another so that voting decisions are determined only by policy and valence.

In general, equation (4) implies that equilibrium strategies will depend on the distribution of centralities. To study the impact of social structure on these strategies, we now transition to considering expected strategies on random graphs.

II. Random Graphs and Social Structure

In this section, we extend results from random graph theory that justify and facilitate our approach. In particular, these techniques allow us to consider centrality on the average graph only, permitting analysis of comparative statics explicitly in terms of social structure—that is, the underlying parameters that govern the social network generative process—rather than of a single realized graph. In the subsequent section, we will employ these results to derive closed-form expressions for the centrality of voters in each group, from which we can recover equilibrium strategies of candidates in terms of group fractionalization, segregation, and density.

We now formalize the concept of an average network in the context of our model, which can be conveniently represented through its average adjacency matrix.

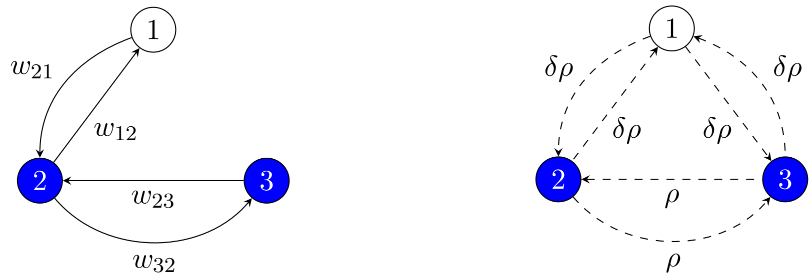
Definition 2. *The average induced adjacency matrix $\bar{\tilde{\mathbf{A}}}^{(n)}$ for a graph of size n is given by, for all $i, j \in \mathcal{V}^{(n)}$,*

$$\bar{\tilde{A}}_{ij}^{(n)} = w_{ij}^{(n)} p_{ij}^{(n)}.$$

where $w_{ij}^{(n)}$ is the weight voter i places on voter j when i considers j a peer and $p_{ij}^{(n)}$ is the probability that voter j is realized as a peer by voter i.[1]

For the following results, we assume that a graph $\mathcal{G}^{(n)} = (\mathcal{V}^{(n)}, \mathcal{E}^{(n)}, w^{(n)})$ of size n, analogously defined as in the previous section, is randomly drawn according to a two-group

[1] Throughout the article, we use the convention that $X^{(n)}$ is an object pertaining to a graph of size n whereas X_n is a sequence of objects with its nth member being $X^{(n)}$.



(A) Fixed realization

(B) Expected

Figure 1. Fixed vs. expected graphs, $w_{ij}p_{ij} = \begin{cases} \rho & \text{if } \ell_i = \ell_j \\ \delta\rho & \text{otherwise} \end{cases}$

stochastic block model with share $s^{(n)} \leq \frac{1}{2}$ of group 1, a probability $p_H^{(n)} \in (0,1)$ of intra-group connection, and a probability $p_L^{(n)} \in (0, p_H^{(n)}]$ of inter-group connection. That is, voters are assumed to be endowed with group membership ex ante and each possible dyad forms a tie independently and randomly with a probability that depends only on whether its members belong to the same group. We further assume for simplicity that for each graph $\mathcal{G}^{(n)}$, we have $w_{ij}^{(n)} = w_H^{(n)}$ for in-group voters and $w_{ij}^{(n)} = w_L^{(n)}$ for out-group voters, with the natural assumption that $w_H^{(n)} \geq w_L^{(n)} > 0$.

Given the generative parameters $\Upsilon^{(n)} = (s^{(n)}, w_L^{(n)}, w_H^{(n)}, p_L^{(n)}, p_H^{(n)})$, we assume that the corresponding sequence of parameters converges, $\Upsilon_n \to \Upsilon$ as $n \to \infty$. Since weights and probabilities do not have separable effects on average, we can re-parameterize the model by letting $\rho := \lim_{n\to\infty} w_H^{(n)} p_H^{(n)}$ and $\delta := \frac{1}{\rho} \lim_{n\to\infty} w_L^{(n)} p_L^{(n)}$, so that $\rho > 0$ captures the baseline propensity to form in-group ties and $\delta \in (0,1)$ reflects the extent of differential propensity to form in-group connections. Naturally, δ^{-1} provides our measure of social segregation (or analogously, homophily).

Figure 1 illustrates the difference between a fixed graph realization and an expected graph in these terms, while Figure 2 highlights the roles of the three main parameters that asymptotically modulate social structure: s governs relative group size and hence fractionalization, δ determines the strengths of between-group ties relative to in-group ties and therefore segregation, and ρ measures within-group tie strength and corresponds to density. In particular, expected density can then be given by $\rho\,[1 - 2s(1-s)(1-\delta)]$ for large n.[2] Here, the first term reflects the effect on density of a simple increase in connection probabilities, while the second reflects the attenuating impact of homophily as the degree of fractionalization changes, impacting the proportion of potential cross-group ties.

[2] The expected density is the expected number of ties within and across each group out of the total number of possible ties on the network, i.e., $\frac{1}{n}(n-1)\left(sn(sn-1)\rho + (1-s)n((1-s)n-1)\rho + 2s(1-s)n^2\delta\rho\right)$. Observing that as $n \to \infty$, $\frac{n}{n-1} \approx 1$, we then have that $\frac{\rho}{n-1}\left(2ns(\delta - \delta s - 1 + s) + n - 1\right) \approx \rho\left(1 - 2s(1-s)(1-\delta)\right)$. Note that density is here defined as the weighted proportion of potential ties that are realized.

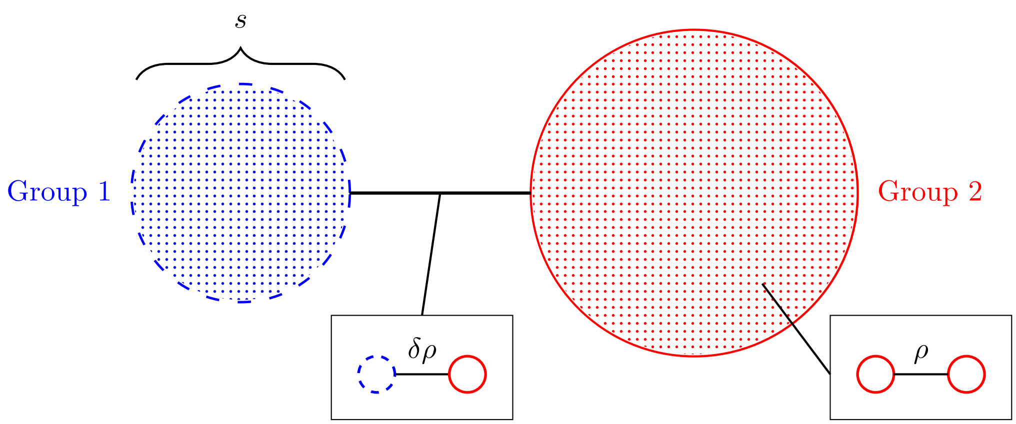


FIGURE 2. SOCIAL STRUCTURE AS ASYMPTOTIC PROPERTIES OF RANDOM GRAPHS

The main result, which draws on the asymptotic bounds on the average adjacency matrix derived in the appendix, allows us to obtain closed-form expressions for each voter's centrality that hold with high probability given large n. In particular, the following lemma allows us to make asymptotic statements that will hold with high probability and therefore justifies the analysis of the expected, rather than the realized, network in the context of large elections. The following assumptions characterize bounds on the kind of society for which the asymptotic approximation can be expected to hold.

Assumption 3. *Take any $\psi > 0$ as given in Lemma 1 and define $d_{min}^{(n)} := \min_i \sum_j w_{ij}^{(n)} p_{ij}^{(n)}$ the minimum expected degree of a voter and $w_{max}^{(n)} := \max_{i,j} w_{ij}^{(n)} > 1$ the greatest weight on any pair of voters in a society of size n. Then for all n, assume that there exists $k(\psi) > 0$ such that $k(\psi) w_{max}^{(n)} \ln(n) < d_{min}^{(n)} < k(\psi)(w_{max}^{(n)})^2 \ln(n)$.*

The key observation here is that the smallest expected degree, and by extension the expected degree of any other voter, must grow at a rate that is at least linear in the natural logarithm of the total population size. For networks in which the degree of voters grows too slowly, the increased sparsity of the resultant network leads to a higher probability of disconnected structures that deviate dramatically from the properties of the average network. At the same time, this degree must not grow too quickly relative to the largest weight, as this can result in networks where excessive spillovers in some components cause behavior that deviates too far from the average network.

Assumption 4. *Given an average induced adjacency matrix $\bar{\tilde{\boldsymbol{A}}}^{(n)}$ and an average weighted degree matrix $\bar{\tilde{\boldsymbol{D}}}^{(n)}$ such that $\{\bar{\tilde{\boldsymbol{D}}}^{(n)}\}_{ii} = \sum_j \bar{\tilde{A}}_{ij}^{(n)}$ and $\{\bar{\tilde{\boldsymbol{D}}}^{(n)}\}_{i \neq j} = 0$ for all n, define the expected normalized Laplacian matrix as $\bar{\tilde{\boldsymbol{L}}}^{(n)} = \boldsymbol{I}_{n \times n} - \bar{\tilde{\boldsymbol{D}}}^{(n)-1/2} \bar{\tilde{\boldsymbol{A}}}^{(n)} \bar{\tilde{\boldsymbol{D}}}^{(n)-1/2}$. Assume that for all n, $\bar{\tilde{\boldsymbol{L}}}^{(n)}$ has a second-smallest eigenvalue bounded away from zero.*

We additionally require a condition on the spectrum of the average Laplacian of the induced adjacency matrix. This guarantees that the network generating model is not too sparse, so that the network is connected with high probability and therefore the solution is always well-defined.

Lemma 1. *Suppose that for a sequence of random graphs $\mathcal{G}_n$, Assumptions 3 and 4 hold. Moreover, denote by $\boldsymbol{c}(\tilde{\boldsymbol{A}}^{(n)})$ the centrality vector of a realized graph $\mathcal{G}^{(n)}$ and $\boldsymbol{c}(\bar{\tilde{\boldsymbol{A}}}^{(n)})$ the corresponding centrality vector of the average graph. Then, for any $\psi > 0$, $\lim_{n\to\infty} \Pr(\|\boldsymbol{c}(\tilde{\boldsymbol{A}}^{(n)}) - \boldsymbol{c}(\bar{\tilde{\boldsymbol{A}}}^{(n)})\| > \psi) = 0$.*

See the appendix for all proofs and related results. Lemma 1 allows us to to restrict our attention to the vector of expected centralities, which is a direct function of parameters, since realized transfers can be expected to be arbitrarily close to these values in large societies. We can now state the following proposition, which provides the basis for our results on social structure.

Proposition 1. *Consider a sequence of random graphs $\mathcal{G}_n$ drawn from a stochastic block model. Under Assumptions 3 and 4, voter centralities can be approximated by a function of parameters $\hat{\mathbf{c}}^{(n)}(s, \delta, \nu, \rho)$ for each graph $\mathcal{G}^{(n)}$. As n grows large, these approximations become arbitrarily close to the true centralities $\boldsymbol{c}$ in any realization: for any $\psi, \eta > 0$, there exists $N > 0$ such that for all $n > N$, $\Pr(|\hat{\mathbf{c}}^{(n)} - \boldsymbol{c}| > \psi) < \eta$. Then, centralities for voters in group 1 and 2 are asymptotically equivalent to, respectively,*

$$
\begin{aligned}
c_1 &\sim \frac{1 - 2\nu\rho(1-s)(1-\delta)}{1 - 2\nu\rho(1 - 2\nu\rho s(1-s)\,(1-\delta^2))} \\
c_2 &\sim \frac{1 - 2\nu\rho s(1-\delta)}{1 - 2\nu\rho(1 - 2\nu\rho s(1-s)\,(1-\delta^2))}.
\end{aligned}
$$

Unlike realized networks, the expected network is necessarily complete, since all voters have positive probability of being connected to all others. Note that this need not apply to any specific realization, as all possible networks on n vertices are in the support of the generative model. Instead, the completeness of the expected network (more precisely, the strict positivity of the matrix of tie formation probabilities) allows us to study how changes in generative parameters affect equilibrium strategies.[3] While it remains possible that realized networks will be drawn in such a way that the equilibrium strategy differs from these expressions, Proposition 1 guarantees that this will occur with vanishing probability in sufficiently large societies.

These centralities can be used to directly recover equilibrium transfers for any large society. Since candidates' optimal transfers differ only to the extent that they possess asymmetric office motivation, we consider without loss of generality the transfer profile offered by candidate 1 (henceforth "candidate") with $\alpha_1 = \alpha$ selecting equilibrium transfer

[3] While the adjacency matrix can be arbitrarily large, it only contains four unique values that correspond to directed connections within and between each group. Because this generates the structure of a block matrix, it is therefore possible to derive an explicit formula for its inverse, which in turn determines the value of each voter's centrality.

profile $\boldsymbol{b}$ in order to retain the focus on the effects of social structure. Let

$$B^{(n)} := \sum_{i \in \mathcal{V}^{(n)}} \left[u'\right]^{-1} \left(\frac{1}{\hat{c}_i^{(n)}} \left[\gamma \left[\hat{\boldsymbol{c}}^{(n)} \cdot \mathbf{1}^{\top} \right] + \frac{1}{\alpha\theta} \right] \right)$$

denote total transfers from the candidate and

$$Q_{ij}^{(n)} := \left(\frac{1}{2} - \frac{[u']^{-1} \left(\frac{1}{\hat{c}_i^{(n)}} \left[\gamma \left[\hat{\boldsymbol{c}}^{(n)} \cdot \mathbf{1}^{\top} \right] + \frac{1}{\alpha\theta} \right] \right)}{\sum_{h=i,j} [u']^{-1} \left(\frac{1}{\hat{c}_h^{(n)}} \left[\gamma \left[\hat{\boldsymbol{c}}^{(n)} \cdot \mathbf{1}^{\top} \right] + \frac{1}{\alpha\theta} \right] \right)} \right)^2$$

denote inequality between voter i and j in any graph $\mathcal{G}^{(n)}$.

We can then characterize $\hat{\boldsymbol{c}} := \lim_{n\to\infty} \hat{\boldsymbol{c}}^{(n)}$ as an explicit function of parameters to examine how the total spending of candidates and the level of inequality between voters depend on social structure. That is, we now consider the limiting levels of expenditure $B := \lim_{n\to\infty} B^{(n)}$ and inequality $Q_{ij} := \lim_{n\to\infty} Q_{ij}^{(n)}$. Because every voter in each group receives the same transfer on average, we can focus on any two voters i' and j' belonging to group 1 and 2, respectively, and denote by $Q = Q_{i'j'}$ the inequality between them. At slight abuse of terminology, we simply refer to the objects B and Q as total transfers and inequality, respectively.

III. Baseline Results

To recover comparative statics, we make several additional assumptions. First, we take voter utility over transfers $u(\cdot)$ to be logarithmic, which is consistent with the more general assumptions in the previous section that guarantee an interior solution. Second, we impose that for each random graph $\mathcal{G}^{(n)}$, $\theta^{(n)} = \frac{\nu}{n}$ for $\nu \in (0, \frac{1}{2})$. This allows ν to reflect the quality of candidate information and satisfies our more general conditions on θ being sufficiently small since the largest eigenvalue is bounded by n and decreasing in n at a rate that guarantees the existence of a finite nonzero asymptotic limit for total spending.

Since we have closed-form expressions for equilibrium strategies, we can take partial derivatives of B and Q with respect to each parameter given our assumptions on $u(\cdot)$. However, these are highly complex objects and cannot be signed by inspection, so we instead solve for boolean assertions on the sign of the derivative using the cylindrical algebraic decomposition algorithm.[4] We additionally verify these computer-algebraic conclusions via numerical computation across a fine grid of the parameter space, coming to identical conclusions. A statement derived with this method is henceforth referred to as a "Fact."

[4] In particular, we use the Mathematica function `Reduce[]`. In each case, the system evaluates to either `TRUE` or `FALSE` for the relevant parameter space.

We begin by stating results that do not pertain to social structure and hold even in the absence of peer effects.

Fact 1. *Candidate office motivations, the voter's valuation for the public good, and the quality of candidate information have the following effects on total transfers and inequality, with* $(+)$ *denoting positive,* $(-)$ *denoting negative, and* (0) *denoting null.*

	office motivation α	*public good* γ	*information* ν
Total transfers B	$+$	$-$	$+$
Inequality Q	0	0	$+$

Greater office motivations encourage candidates to divert public resources towards private provision, while a stronger value for the public good reduces this incentive for diversion. Moreover, as better information increases the marginal value of targeting any voter, it incentivizes spending and raises the candidate's sensitivity to other incentives.

Inequality, on the other hand, is consistently unaffected by changes in these parameters, as candidate beliefs about optimal allocation of transfers across groups do not change in their office motivation or the voters' value for the public good.

With regard to social structure, the difference in the expected number of connections of members of each group shifts candidates' targeting incentives in a way that reverberates across the network. For the following results, we assume without loss of generality that group 1 is the minority, $s < \frac{1}{2}$, so that fractionalization is increasing in s.

Fact 2. *Fractionalization, segregation, and in-group connection probabilities have the following effects on total transfers and inequality, with* $(+)$ *denoting positive and* $(-)$ *denoting negative.*

	fractionalization s	*segregation* δ^{-1}	*in-group ties* ρ
Total transfers B	$-$	$-$	$+$
Inequality Q	$-$	$+$	$+$

As society becomes less segregated and less fractionalized (i.e., more dominated by a single large majority), candidates face an incentive to divert expenditure. To see why this is the case, recall that with $\delta < 1$, relabeling a single voter from the majority to the minority group results in a decrease in the average degree and thus total influence of members of the majority group, which is now marginally smaller, but also results in a corresponding increase in influence for members of the now-larger minority group. At the same time, the total weight of all intergroup or "weak" ties $n^2 s(1-s)\delta\rho$ increases as the group sizes become more even.

There are thus four basic mechanisms that connect an increase in the majority group's share to candidate incentives. First, transfers to the majority group increase in marginal value, which encourages greater expenditure. Second, the opposite holds true for the minority group. Third, both of these effects are attenuated by their mirror opposite: the penalty incurred for diversion of public goods is increased. Finally, the effect of transfers to *any* voter has an increased impact on spillovers to the other group. It is the first effect that dominates regardless of the actual level of homophily, implying that the social structure that most favors private transfers is one in which society consists only of a single group.

A similar logic drives the impact of segregation and density on total spending. Holding the baseline connection probability ρ constant, an increase in δ (lower segregation) implies an increase in all cross-group tie probabilities. As a consequence, the net influence of *every* voter on all others is greater, which directly incentivizes more spending—although this effect is again attenuated by γ. Similarly, for any amount of segregation, an increase in ρ makes the entire network more dense in expectation and therefore results in increased social influence and hence more expenditure.

While, as just discussed, the fact that all voters are interconnected implies that changes in social structure tend to increase or decrease the value of all voters at once, these effects are not necessarily symmetric across groups. In general, members of the majority group are always more valuable, since their higher expected degrees are associated with a greater total influence per voter. Any change in parameters that results in a relatively higher marginal gain for minority members than for majority members will then lead to a reduction in inequality.

Reductions in segregation are associated with a greater total transfers, but also reduce inequality. While the value of a transfer to any voter increases as the proportion of cross-group ties goes up, this benefit accrues disproportionately to members of the minority group. This is because each new cross-group tie improves minority voters' ability to influence members of the larger group, which in turn brings greater electoral returns. While majorities similarly benefit from the ability to influence minorities, the smaller size of the minority means that these corresponding benefits are smaller.

Fractionalization (i.e., increases in the size of the minority group) also benefit voters in the minority group and reduce inequality. First, a growing minority group rises the relative value of each of its members due to the direct effect of raising their expected degrees. At the same time, more equal group sizes increase the number of cross-group ties, simultaneously allowing minority members greater influence over the shrinking majority.

Unlike these two parameters, increases in the baseline connection probability ρ accentuate inequality, benefiting majority members to a greater extent. This is because, holding δ constant, an increase of $\Delta\rho$ has a Δ impact on within group ties (which are more numerous among the majority), but only a $\Delta\delta$ effect on cross-group ties, which bear the most importance for the minority. As a consequence, dense networks will tend to see the most expenditure, but also the greatest inequality in targeted transfers, as overall density tends to benefit members of the majority more.

IV. Heterogeneous Information

In the preceding section, we consider the case where candidates possess no inherent incentive for transfers to either group beyond its size. In this section, we study how the mechanisms highlighted so far interact with the information available to candidates regarding voters in each group.

Specifically, we allow the precision of a candidate's information to vary systematically across voters. In particular, we focus on variation in information that arises due to systematic differences between the two groups. For instance, if the effect of group identity on voting is stronger in one party than another, candidates may view those associated with the "weaker" group as more likely to be swing voters, as their vote choice exhibits higher variance conditional on their group label. Intuitively, the first-order effect of this variation is to reduce the value of transfers to members of the less predictable group. Nevertheless, it is unclear a priori how this affects the comparative statics derived in the baseline model, as the reduced value of members of this group also reduces the significance of all flow-on effects that go through them in the network.

There are two distinct incentives for candidates that compete. First, candidates value members of the majority more highly due to their greater ability to influence. However, voters in the more predictable group are now both more attractive to target individually and have a more reliable impact on their neighbors. How does social structure modulate the relative importance of these two channels?

A. Equilibrium with Heterogeneous Information

We begin from the setup of the baseline model, with the distinction that the information held by a candidate about voter i's preferences is allowed to vary. In particular, voter i's net preference for the candidate, ε_i, is now drawn from one of two uniform distributions with density parameter $\theta_i \in \{\underline{\theta}^{(n)}, \bar{\theta}^{(n)}\}$ with $\bar{\theta}^{(n)} > \underline{\theta}^{(n)}$ for any graph of size n. We can think of θ_i as voter i's private type, which is unknown to candidates. While the candidate does not know which distribution voter i's net preference was drawn from, they receive signals about each voter's type $m_i \in \{\underline{\theta}^{(n)}, \bar{\theta}^{(n)}\}$ such that $m_i = \theta_i$ with a probability greater than half that depends on the voter. In other words, the candidate receives informative signals about the preferences of voters and those signals may be more precise for some voters than for others. After receiving signals $\boldsymbol{m}_k = (m_{1k}, \ldots, m_{nk})$, each candidate k forms posterior beliefs $\boldsymbol{\mu}_k = (\mu_{1k}, \ldots, \mu_{nk})$ where $\mu_{ik} := \Pr(\theta_i = \underline{\theta}^{(n)} | m_{ik})$ and distribute payments according to expected types $\hat{\theta}_{ik} := \mathbb{E}_{\mu_k}[\theta_i]$.

With uncertainty over voter types, we can write candidate k's problem as

$$\text{(P2)} \qquad \max_{\boldsymbol{b}_k} \ \alpha_k \sum_{i \in \mathcal{V}} \mathbb{E}_{\mu_k}[\phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k})] - \boldsymbol{b}_k \cdot \mathbf{1}^\top$$

$$\text{subject to} \ \ b_{ik} \geq 0 \text{ for all } i.$$

Exactly as before, each voter's probability of voting for candidate 1 is a function of all other vote probabilities. Unlike the baseline model, however, candidates maximize an expected utility that now depends on their posterior beliefs of voter types. In particular, we need to characterize the candidates' expected vote share conditional on their signals. Letting $\phi_i = \phi_{i1}(\boldsymbol{b})$ as before without loss of generality, these can be expressed for candidate 1 as

$$\mathbb{E}_{\mu_1}[\phi_i] = \frac{1}{2} + \mathbb{E}_{\mu_1}[\theta_i]\tilde{U}_{i1} + \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}(2\mathbb{E}_{\mu_1}[\theta_i\phi_j] - \mathbb{E}_{\mu_1}[\theta_i])$$

where $\tilde{U}_{i1} := (-1)^{x_i - 1} + u(b_{i1}) - \mathbb{E}_{\mu_1}[u(b_{i2})] + \gamma \sum_m (\mathbb{E}_{\mu_1}[b_{m2}] - b_{m1})$, and analogously for candidate 2. Additionally, note that $\mathbb{E}_{\mu_k}[\theta_i] = \bar{\theta} - \mu_{ik}(\bar{\theta} - \underline{\theta})$; this can be thought of as a candidate k's net information about voter i, taking into account both first-order uncertainty about i's vote choice and second-order uncertainty over her type. To solve for equilibrium transfers, we first make the following assumption.

Assumption 5. *Denote by $\mu^{max} := \max_{ik} \mu_{ik}$ and $\mu^{min} := \min_{ik} \mu_{ik}$ the largest and smallest posteriors, respectively, with $\hat{\theta}^{max}$ and $\hat{\theta}^{min}$ as the corresponding expectations. Then, for any sequence of graphs $\mathcal{G}_n$, let there be corresponding sequences of largest and smallest posteriors, $(\mu^{max})_n$ and $(\mu^{min})_n$, such that there exists a constant rate of information decay, $\chi := \frac{1}{(\hat{\theta}^{max})^{(n)} n} \in (2, \infty)$, and a constant relative informational disadvantage for the minimum voter, $\nu := \frac{(\hat{\theta}^{min})^{(n)}}{(\hat{\theta}^{max})^{(n)}} \in (0,1)$, for each element graph $\mathcal{G}^{(n)}$.*

This assumption ensures two things. First, the constant rate of information decay guarantees that the maximum quality of information for every element graph is low enough so that the solution is always well-defined. Second, the constant relative informational disadvantage to the minimum voter serves to bound the divergence in information quality between any two voters across element graphs.

Whereas ν in the baseline model reflects overall quality of candidate information about all voters, ν in this section reflects the average quality of information over the least predictable voter. On the other hand, the new parameter χ governs (the inverse of) the average quality of information over the most predictable voter.

Given this new assumption, we can state the following.

Lemma 2. $\lim_{n \to \infty} \mathbb{E}_{\mu_k}[\theta_i \phi_j] = \left(\underline{\theta} - \mu_{ik}\left(\bar{\theta} - \underline{\theta}\right)\right) \mathbb{E}_{\mu_k}[\phi_j]$ *for any $i, j \in \mathcal{V}$ and any $\boldsymbol{\mu}_k$.*

This lemma enables us to disentangle the effect of changes in a particular voter's type from the influence it has on the vote probability of another voter as the network grows sufficiently large. In particular, a voter i's type θ_i and a voter j's vote probability ϕ_j become asymptotically independent. Using this fact, we can write the optimality condition in a similar way as before,

$$(\boldsymbol{J}[\boldsymbol{u}] - \boldsymbol{\Gamma})^\top \cdot \left(\boldsymbol{I} - 2\Theta_k \tilde{\boldsymbol{A}}\right)^{-1} \cdot \mathbf{1} = \Theta_k^{-1} \cdot \frac{(1-\boldsymbol{\lambda}^*)}{\alpha_k},$$

where $\Theta_k := \bar{\theta}\boldsymbol{I} - (\bar{\theta} - \underline{\theta})\boldsymbol{M}_k$ and $\boldsymbol{M}_k$ is an $n \times n$ diagonal matrix with posteriors μ_{ik} as nonzero elements, and the remaining objects are as defined in the baseline. From this, we can express equilibrium transfers by

$$b_{ik} = \left[u'\right]^{-1}\left(\frac{1}{c_{ik}(\boldsymbol{w}, \underline{\theta}, \bar{\theta}; \mathcal{G}, \boldsymbol{\mu}_k)}\left[\gamma C_k(\boldsymbol{w}, \underline{\theta}, \bar{\theta}; \mathcal{G}, \boldsymbol{\mu}_k) + \frac{1}{\alpha \hat{\theta}_{ik}}\right]\right),$$

where the only difference from the previous section is that payments depend on beliefs about voter i's type, $\hat{\theta}_{ik} := \mathbb{E}_{\mu_k}[\theta_i]$, which also influences the centrality measure c_{ik}. Now, c_{ik} is the ith element of $\boldsymbol{c}_k = (\boldsymbol{I} - 2\Theta_k \tilde{\boldsymbol{A}})^{-1} \cdot \mathbf{1}$ with $C_k(\cdot) := \sum_i c_{ik}$. Since one candidate's transfers do not affect the optimal transfer profile for the other, both candidates will act according to their own beliefs in equilibrium. Hence, to study the impact of parameters on equilibrium play, it is sufficient to consider a single candidate with arbitrary posteriors that satisfy Assumption 5.

B. Heterogeneous Information Results

To recover comparative statics with heterogeneous information, we retain the assumption of logarithmic voter utility over transfers. We also now allow for group 1 to be any size $s \in (0, 1)$ to focus on informational disparities. To ensure that Assumption 2 holds, we impose that voters of the same group have the same expected type from the perspective of the candidate such that, for each random graph $\mathcal{G}^{(n)}$, $\hat{\theta}_1^{(n)} = \frac{1}{\chi n}$ for $\chi \in (2, \infty)$ and $\hat{\theta}_2^{(n)} = \nu \hat{\theta}_1^{(n)}$ for $\nu \in (0, 1)$ where 1 and 2 refer to voter groups. This setup allows us to focus on the effect of relative changes in the quality of information about individuals in one group over the other. In particular, $\nu < 1$ implies without loss of generality that group 1 has superior information on average, with $\nu = 1$ implying equal quality. Note that as $\nu \to 1$, the total information available to the candidate across the whole network also increases.

Proposition 2. *Consider a sequence of random graphs $\mathcal{G}_n$ drawn from a stochastic block model. Under the assumptions of Proposition 1 and Assumption 5, expected voter centralities given candidate k's information can be approximated by a function of parameters $\hat{\boldsymbol{c}}_k^{(n)}(s, \delta, \nu, \rho, \chi)$ for each graph $\mathcal{G}^{(n)}$. These approximations become arbitrarily close to the true centralities $\boldsymbol{c}$ in any realization as $n \to \infty$. Then, centralities for voters in group 1* *and 2 are asymptotically equivalent to, respectively,*

$$c_1 \sim \frac{\chi^2 - 4\rho\chi(1-s)\,(\nu-\delta)}{\chi^2 + 4\rho\,(4\rho\nu s(1-s)\,(1-\delta^2) - \chi\,(\nu + s\,(1+\nu)))}$$
$$c_2 \sim \frac{\chi^2 - 4\rho\chi s(1-\delta\nu)}{\chi^2 + 4\rho\,(4\rho\nu s(1-s)\,(1-\delta^2) - \chi(\nu + s(1+\nu)))},$$

As in the baseline model, equilibrium strategies may differ from those implied by these expressions in any particular network realization; however, Proposition 1 ensures that this occurs with vanishing probability as the number of voters grows sufficiently large.

Disconnected Model.—It is instructive to briefly explore the disconnected case with heterogeneous information. Candidate strategies straightforwardly reduce to

$$b_{ik} = \left[u'\right]^{-1}\left(\frac{\gamma\alpha \cdot \mathrm{tr}(\Theta_k) + 1}{\alpha\hat{\theta}_{ik}}\right) \tag{6}$$

where $\mathrm{tr}(\Theta_k) = \sum_i \hat{\theta}_{ik}$ denotes the trace of Θ_k.

Although this expression is almost identical to the disconnected solution to the baseline model in expression (5), there is one important difference: the total information Θ_k now appears in the numerator instead of simply n. The immediate consequence of this is that, while an increase in office motivation is still associated with greater spending, the relative impact of α and γ on group-level transfers is now contingent on the informational ratio of each voter, $\hat{\theta}_{ik}/\mathrm{tr}(\Theta_k)$. This fact has important consequences for redistributive politics even in the presence of network effects, as the following section demonstrates.

In the absence of peer effects, we can recover total transfers and inequality as

$$B = \frac{\alpha((\alpha\gamma + \chi)\nu + s(1-\nu)\chi)}{(\alpha\gamma + \chi)(\alpha\gamma\nu + \chi)} \qquad \textit{and} \qquad Q = \frac{((1-\nu)\chi)^2}{4((2\alpha\gamma + \chi)\nu + \chi)^2},$$

respectively. As in the baseline model, neither density nor segregation has an impact on transfers, since these govern relationships between voters. However, social structure still plays a role through group share. The imbalance in the value of transfers made to members of group 1 implies that spending will also increase as a greater proportion of the population belongs to this group. The magnitude of this effect is also increasing in the extent of informational heterogeneity $(1-\nu)\chi$, as spending becomes more sensitive to changes in group composition the more pronounced the imbalance.

A second consequence of introducing heterogeneity in information is that members of the more predictable group now receive systematically higher transfers, in contrast to the disconnected case of the baseline model where inequality is constant in parameters. This basic mechanism is unsurprising; however, it is noteworthy that this channel also generates a new connection between inequality and the value of the public good, reflected in α and γ. Specifically, as the public good becomes more valuable to voters (increasing γ) or less valuable to the candidate (increasing α), the informational advantage of group 1 is attenuated.

The reason for this difference lies in the varying trade-off faced by increasingly office-motivated candidates between maximizing the value of transfers to each individual and minimizing the total cost to all voters of diverting resources, exemplified in expressions (5) and (6). Hence, as γ increases and the penalty to the candidate for offering transfers grows larger, the equilibrium transfer to members of the predictable group are reduced at a faster rate, reducing inequality. Greater certainty over these voters' preferences yields greater marginal returns from diverting resources away from them.

Conversely, increases in office motivation α result in greater transfers to members of the more predictable group, which in turn exacerbates inequality. Because the candidate has a stronger incentive to make transfers in general, the benefits accrue disproportionately to those voters who are deemed most likely to change their votes.

Connected Model.—We now return to the connected model with peer effects. It is no longer useful to explore the effect of parameters on total transfers and inequality using partial derivatives, as the added complexity render the conditions analytically uninformative. Nonetheless, we can characterize sharp conditions in which group 1 voters receive greater transfers than those in group 2. This allows us to understand the effects of social structure on group-level biases in this setting before turning to a numerical analysis of total transfers.

This set of results is derived by solving for boolean assertions using the cylindrical algebraic decomposition algorithm, as previously described. A statement recovered in this way continues to be called a "Fact."

Fact 3. *Group 1 voters receive larger transfers than those in group 2 if the more predictable group is sufficiently large, $s > s^* := \frac{\nu-\delta}{(1-\delta)(1+\nu)}$ or segregation is sufficiently low, $\delta \geq \nu$.*

A key difference between the baseline model and the case of heterogeneous information relates to the role of social structure. In the baseline model, transfers decrease in both fractionalization and segregation. Now, the effect of social structure is conditional on its relationship to the information structure. In general, the equilibrium transfer profile depends on the balance of two competing effects: an information effect determined by ν and χ, which favors the more predictable group, and a network effect governed by δ and ρ, which favors the larger and more connected group.

The balance of these two effects is strongly related to group share, which affects both the size of the network effect and the proportion of more valuable voters on the network. When group 1 is in the majority, we observe the same pattern as before: fractionalization decreases expenditure. This is because the information and network effects point in the same direction, with increases in the share of more valuable group 1 members encouraging greater spending.

When the more predictable group 1 is small—more precisely, when $s < s^*$—these two mechanisms push candidates in opposite directions. An increase in the more predictable group's share leads to a higher proportion of high-value voters, but also reduces the size of the more connected majority group as in the baseline model. The threshold at which this happens depends on both segregation δ and information asymmetry ν. The more equal the two groups are (high ν), the higher this threshold is, as the reduced informational advantage to group 1 results in a more dominant network effect.

Conversely, low segregation (high δ) tends to push the threshold down, with the network effect becoming weaker as the two groups become more connected, resulting in the candidate favoring voters in the predictable group for a wider range of group fractionalization. This same logic implies that group 1 will always be favored when δ exceeds ν, as this guarantees that the information effect will always dominate.

Moreover, the information effect may continue to dominate the network effect even if the size of group 1 is small $s \leq s^*$ or segregation is high $\delta < \nu$.

Fact 4. *Assume that the conditions of Fact 3 do not hold. Then, group 1 voters can still receive larger transfers than those in group 2 when candidate office motivation is sufficiently low, $\alpha < \alpha^*(s, \delta, \rho, \gamma, \nu, \chi)$.*[5] *In particular, this occurs when*

(i) information disparities are sufficiently large, $\nu \leq \frac{1}{2}$;

(ii) segregation is sufficiently low, $\delta \geq 2\nu - 1$;

(iii) candidates are sufficiently ill-informed in general, $\chi \geq \chi^* := \frac{2(\nu(1-s(2-\delta\nu))-\delta(1-s))}{1-\nu}$;

(iv) the more predictable group is sufficiently large, $s \geq s^{**} := \frac{1+\delta-2\nu}{\delta(1+\nu^2)-2\nu}$; or

(v) in-group connection probabilities are sufficiently weak, $\rho < \rho^* := \frac{\chi}{\chi^*}$.

Fact 4 corresponds to network and information effects that are in close opposition. With $s \leq s^*$ and $\delta < \nu$, group 1 is sufficiently small not to benefit from greater network spillovers, yet possesses a large enough informational advantage that it could in principle overcome this. Unlike in the baseline model, where α simply scales equilibrium transfers, the introduction of this trade off now leads to distributional consequences for office motivation. While the information and network effects are in opposition, they differ in their returns to scale. Although shifts in θ affect a candidate's predictions regarding network spillovers, their primary effect is to shift the ex-ante value of voters in each group, which impacts their electoral returns linearly.

[5] The full expression for $\alpha^*(\cdot)$ is omitted due to its size and that it does not provide additional clarity. It is available for review in our replication materials.

On the other hand, network spillovers operate exponentially, with marginal increases to the probability of a voter's support affecting everyone else, which in turn further influences the original voter. As candidates become more office motivated, they are willing to divert more of the public good to recover electoral support, which magnifies these exponential gains from the network effect. When $\alpha > \alpha^*$, therefore, candidates are willing to extend enough transfers that the expected returns from more predictable voters in group 1 must always be less than those from more connected group 2 voters, regardless of the informational balance. Notably, this threshold is a function of all of the remaining parameters, as these determine the point at which exponential increases from network spillovers are sufficiently strong as to outweigh the additional gains group 1 voters offer from better information.

When $\alpha < \alpha^*$, there exist parameter combinations such that the information effect can outweigh the network effect. First, pertaining to conditions (i) and (ii) of Fact 4, the information effect may prevail when group 1's information advantage is large enough and segregation is low enough. The more segregated a society, the greater the peer effects, reducing the value of greater predictability. Specifically, if the amount of segregation δ does not exceed the threshold value of $2\nu - 1$, then the disadvantage of being a minority group is insufficient to outweigh the informational superiority of group 1. Consequently, if the magnitude of the informational asymmetry is more than twofold, it is impossible for segregation to be so extreme as to outweigh the information effect.

Another way group 1 voters may receive higher transfers is a decrease in the overall information available to the candidate, corresponding to on condition (iii) of the proposition: $\chi \geq \chi^*$ holding ν constant. This result is somewhat counterintuitive. The benefit of additional information accrues disproportionately to group 1 voters, since ν attenuates the information available to group 2. Due to the concavity of $u(\cdot)$, the transfer profile tends towards equality as the total information on the graph increases. Notably, the threshold additionally depends on δ and s, since the benefit to group 2 voters of information is mediated by their size and connection to the smaller group.

Pertaining to condition (iv), a lower threshold s^{**} exists in addition to the threshold s^*. Like s^*, this lower threshold depends on ν and δ, and determines the group size at which the information effect and network effect offset one another. However, unlike the higher threshold, which determines the point at which this occurs for highly office-motivated candidates for whom the exponential network spillovers outweigh the benefit of more information, s^{**} determines the point at which these equalize for candidates that place comparable weight on electoral gains and public goods provision. Notably, α^* also depends on s, so that three possibilities exist: the two effects equalize at either (i) s^{**} when α is small; (ii) some $s' \in (s^{**}, s^*)$ such that $\alpha = \alpha^*(s'; \cdot)$ when α is intermediate; or (iii) s^* when α is large.

The final condition (v) is when the network is highly segregated and candidates have good information overall, but a significant informational asymmetry persists. In this case,

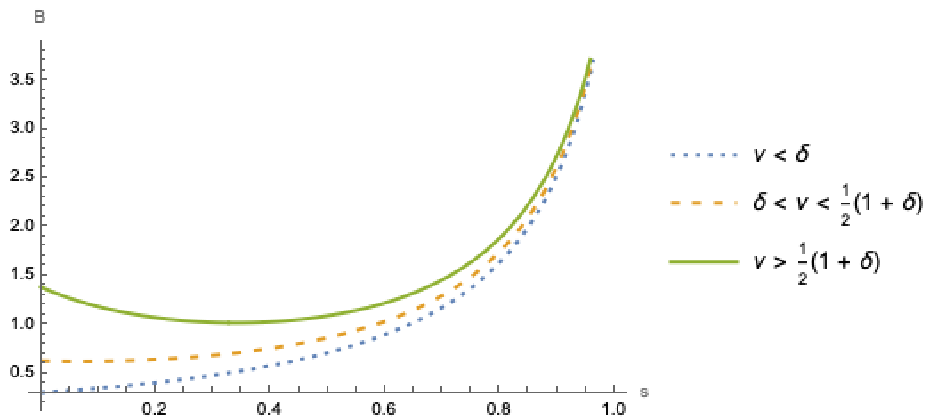


Figure 3. Effect of group share on total transfers under varying information asymmetry

both effects are quite weak: the network is weakly connected, but the overall information available is high, attenuating group 1's informational advantage given the tendency towards equality as candidates become highly informed. For the information effect to win out, group 2's size advantage must not be too large. Moreover, both groups must be weakly connected within themselves, not only to one another. In fact, this threshold corresponds precisely to the fraction of information held by the candidate relative to the threshold amount, $\frac{\chi}{\chi^*}$.

Together, Facts 3 and 4 provide the complete set of conditions for more predictable group 1 voters to receive higher transfers. While these results provide a precise characterization of the relative transfers afforded to members of each group, they do not directly relate to the candidate's total expenditure. Indeed, as noted above, the expression for B yields intractable derivatives. Nevertheless, the thresholds discussed closely relate to general patterns in expenditure, which we now represent graphically.

Figure 3 shows the impact of varying ν and δ, which directly modulate the relative strength of the information and network effects on the relationship between group size and total transfers. When ν is small relative to δ, the information effect dominates, leading to total expenditure being dominated almost entirely by the share of group 1. Intuitively, as more of the more valuable voters are added to the network, the candidate is driven to spend more. Notably, the growth in expenditure increases to exponential as group 1 moves into the majority, as both effects now work in concert to drive up spending.

When ν exceeds the $\frac{1}{2}(1+\delta)$ threshold, however, increases in the size of group 1 can decrease spending. This occurs as a result of the shift from targeting group 2 to group 1 voters. In the baseline, when s is very small, the network advantage for group 2 is overwhelming and, as s increases so that group 1 and 2 converge in size, the candidate extends less transfers in total. This effect is still present in the heterogeneous case, however, spending will now also respond to informational advantages. Consequently, the size of group 1 at which total transfers are minimal is interior but below the halfway point.

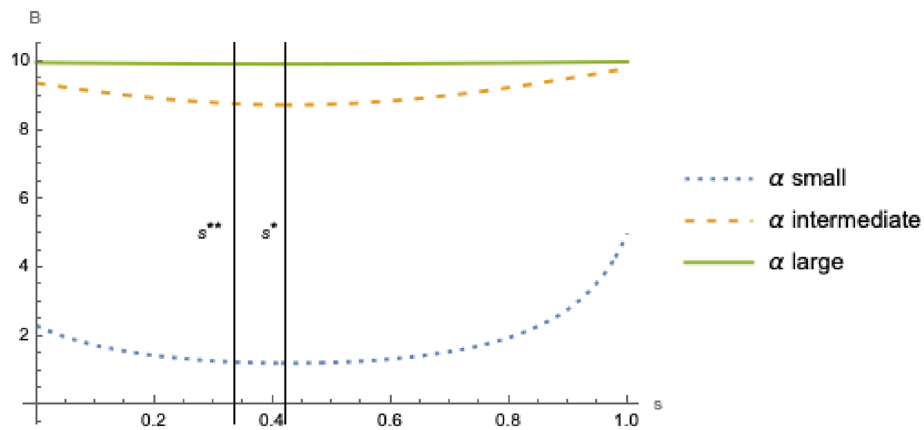


FIGURE 4. EFFECT OF GROUP SHARE ON TOTAL TRANSFERS UNDER VARYING OFFICE MOTIVATION

On the other hand, Figure 4 illustrates the impact of office motivation holding ν and δ constant. Here, the thresholds s^* and s^{**} are marked to reflect the cutpoints at which the candidate shifts from targeting group 2 voters to those in group 1. While this yields the same general pattern as noted above—the network effect dominates for small s until the two equalize at either s^{**}, s^*, or some $s' \in (s^{**}, s^*)$, depending on whether office motivations are small, large, or intermediate—candidate office motivations also flatten the curve. This is rooted in the logic of exponential returns to scale from network effects already discussed: sufficiently office-motivated candidates are driven to extend larger individual transfers in equilibrium, increasing the effect of peer influence on vote probabilities and making total transfers relatively unresponsive to changes in group share.

V. Conclusion

In this article, we find that the incorporation of peer effects raises the marginal value of transfers to central voters, resulting in a diversion of public resources that may greatly exceed the amount voters would otherwise prefer. In our baseline that centers on the role of network effects, candidates prioritize voters in the majority group and extend the fewest transfers when groups are equipopulous and segregated.

When we extend the model to consider candidates with heterogeneous information about the electorate, information can overcome the tyranny of the majority caused by the network effects. In particular, this will occur when groups are integrated, as greater inter-group connections diminishes the extent to which voters in the less predictable group can be advantaged in their ability to influence others, causing candidates to prefer more predictable voters even if they are in the minority group. This may also occur if candidates are low in their office motivations, as a reduced incentive to extend transfers causes the candidates to be more selective when taking resources away from the public good. This will especially be true if peer relations are weak or the overall quality of information about voters is poor, in which case candidates may prefer to target minority voters.

These results suggest several new insights. First, intensive targeted redistribution will be more likely to occur in social contexts that are maximally homogeneous, with both minimum fractionalization (i.e., almost all voters belonging to one group) and low levels of social segregation. This has broad implications on the way in which we might expect political institutions to vary across different societies. In contexts where group boundaries are hard and salient, targeted redistributional strategies are inhibited by the lack of positive cross-group externalities, resulting in competition centering on the promise of public good provision. In contrast, politicians have a greater incentive to redistribute in highly homogeneous societies with weak group divisions or a single dominant group, leading to a greater focus on targeted redistribution. Notably, this mechanism sheds some light on the contradictory findings regarding the effect of diversity on public goods provision in localities of varying size (Schaeffer, 2013; Singh and Vom Hau, 2016): although small societies can produce either relation, our results suggest that diversity favors public goods provision in large societies because it reduces the ability of politicians to exploit peer effects for electoral advantage.

Second, members of the majority group are likely to benefit disproportionately from private provision, especially when candidates are well-informed about voters. In the absence of strong in-group preferences or informational advantages, this will tend to occur regardless of the group affiliation of those dispensing resources and may lead to targeting voters that ex-ante prefer the opposition. This occurs for the same reason transfers become more targeted as society becomes more homogeneous: the primary benefit to targeting individuals is the potential to exploit peer effects and sway many voters simultaneously. Because social connections are more numerous in larger groups, we expect minorities to be systematically disadvantaged the smaller and more isolated they become. Even candidates belonging to their own group face a strong incentive to target the more electorally profitable majority group and provide fewer public goods than otherwise.

Third, information affects the role of social structure in redistributive politics and vice versa. When one group is more predictable than the other, this creates an incentive for candidates to target voters whose preferences they know more about. When the group associated with more information is also relatively small, the effect of information is in tension with the effect of the network, which favors voters in the larger group as a result of their greater ability to influence the votes of others. Our model suggests that the information effect dominates when either the information imbalance is larger than the level of cross-group integration, or when candidates are weakly office motivated and poorly informed relative to the strength of peer effects.

This is particularly notable with regard to candidates who themselves belong to political minorities and hence possess more certain information about the preferences of in-group members. For instance, this dilemma may arise for members of fringe parties catering to a tightly-knit economic or ideological community, or ethnic minority candidates seeking national representation. Our model suggests that such candidates will engage in favoritism—i.e., promising transfers to members of their own group—only if their group is strongly integrated into the majority group or if their motivation to hold office is not very strong. First, as the minority group becomes more segregated from the majority group, the reduced ability of the candidate to sway many voters simultaneously through transfers to minorities causes them to allocate more to voters in the majority group instead. Second, as minority candidates care more about holding office, they will instead seek to induce members of the other, larger group with targeted transfers, driven by the allure of social spillovers despite knowing less about individual voters.

Finally, our results raise several questions about social structure and redistributive politics. Our setting explores politicians that are endowed with heterogeneous information about voters; however, how might electoral strategies change if learning is costly? Specifically, would the role of social structure change if a candidate's cost of learning about a voter depends on their positions in the social network? Moreover, we take network formation as exogenous to the strategic interaction at hand. How might voters choose their friends if they were aware that it might impact their subsequent welfare? These are interesting questions for future research.

VI. Appendix

In the current section, we begin by presenting several additional results that facilitate our analysis, alongside their corresponding proofs. Thereafter, we present the proofs for statements presented in the main text. Refer to the accompanying replication materials for additional details.

First, we state a concentration inequality for sums of independent random matrices with bounded operator norms that forms the basis of our asymptotic bounds. Analogous to Chernoff bounds for scalar-valued random variables, this bound exploits the sub-additivity of matrix cumulant-generating functions together with the Laplace transform method to guarantee that the sum is not too far from its expectation. The key idea in our context is that, despite the presence of general interdependencies in the network, the independence of each individual tie under the stochastic block model guarantees that the adjacency matrix can be decomposed into independent random matrices.

Lemma A1 (Matrix Bernstein Inequality)**.** *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ be independent random $n \times n$ Hermitian matrices and set $M > 0 : ||\boldsymbol{X}_i - \mathbb{E}(\boldsymbol{X}_i)||_2 \leq M$ for all $i = 1, \ldots, m$. Then for any $a > 0$,*

$$\Pr(||\boldsymbol{X} - \mathbb{E}(\boldsymbol{X})||_2 > a) \leq 2n \exp\left(-\frac{a^2}{2v^2 + 2Ma/3}\right)$$

where $\boldsymbol{X} = \sum_{i=1}^{m} \boldsymbol{X}_i$ *and* $v^2 = || \sum_{i=1}^{m} \mathbb{V}(\boldsymbol{X}_i) ||$.

This is the standard Bernstein Inequality for matrices. See, for example, Theorem 5 in Chung and Radcliffe (2011) for the proof.

Next, we derive a Chernoff bound for sums of weighted Bernoulli random variables. In addition to individual connections, it is also necessary to provide bounds on the degree of each voter, which is precisely such a sum. Hence, by assuming that all weights are bounded above, we are able to calculate limits on the probability of deviation of voter degrees from their expected degree.

Lemma A2 (Chernoff Bound for Weighted Bernoulli Sums)**.** *Let* $X_1, \dots, X_m$ *be independent random variables distributed* $X_i \sim w_i \cdot \text{Bernoulli}(p_i)$ *and define* $\Delta := \sum_i w_i p_i$ *and* $\omega := \max_i w_i$. *Then for any* $0 < t < 1$*,*

$$\Pr(|X - \Delta| > t\Delta) \leq 2 \exp\left(\frac{-\Delta}{3\omega} (t^2 - 3) \right)$$

where $X = \sum_{i=1}^{m} X_i$.

Proof. We proceed by bounding the moment generation function (MGF) of each X_i. First, denote by $M_Y(s) = \mathbb{E}[e^{sY}]$ the MGF of Y. Then,

$$M_{X_i}(s) = 1 + p_i(e^{sw_i} - 1) \leq e^{p_i(e^{sw_i} - 1)}$$

and hence

$$M_X(s) = \prod_i M_{X_i}(s) \leq e^{\Delta(e^{s\omega} - 1)}.$$

Next, from Markov's inequality we have

$$\begin{aligned} \Pr(X \geq (1+t)\Delta) &\leq \frac{e^{\Delta(e^{s\omega} - 1)}}{e^{s(1+t)\Delta}} \\ &= \left(\frac{\omega e^{1+t-\omega}}{(1+t)^{1+t}} \right)^{\frac{\Delta}{\omega}}, \end{aligned}$$

where we set $s = \frac{1}{\omega} \ln\left(\frac{1+t}{\omega}\right)$ to minimize the bound. Then taking the logarithm of the right-hand side gives

$$\begin{aligned}
\ln\left(\left(\frac{\omega e^{1+t-\omega}}{(1+t)^{1+t}}\right)^{\frac{\Delta}{\omega}}\right) &= \frac{\Delta}{\omega}\left(1+t-\omega+\ln(\omega)-(1+t)\ln(1+t)\right) \\
&\leq \frac{\Delta}{\omega}\left(1+t-\frac{2t(1+t)}{2+t}\right) \\
&= -\frac{\Delta}{\omega}\left(\frac{(t-2)(t+1)}{2+t}\right),
\end{aligned}$$

because $\ln(1+x) \geq \frac{x}{1+x/2}$ for $x > 0$. Hence, we can write

$$\text{(A1)} \qquad \Pr(X-\Delta \geq t\Delta) \leq \exp\left(-\frac{\Delta}{\omega}\left(\frac{(t-2)(t+1)}{2+t}\right)\right).$$

Moreover, an analogous argument with $s = \frac{1}{\omega}\ln\left(\frac{1-t}{\omega}\right)$ and applying $\ln(1-x) \geq \frac{x^2}{2} - x$ for $0 < x < 1$ yields the lower tail bound of

$$\text{(A2)} \qquad \Pr(X-\Delta \leq -t\Delta) \leq \exp\left(-\frac{\Delta}{2\omega}\left(t^2-2\right)\right).$$

Putting together equations (A1) and (A2) yields, for $0 < t < 1$,

$$\begin{aligned}
\Pr(|X-\Delta| \geq t\Delta) &= \Pr(X-\Delta \geq t\Delta) + \Pr(X-\Delta \leq -t\Delta) \\
&\leq \exp\left(-\frac{\Delta}{\omega}\left(\frac{(t-2)(t+1)}{2+t}\right)\right) + \exp\left(-\frac{\Delta}{2\omega}\left(t^2-2\right)\right) \\
&\leq \exp\left(-\frac{\Delta}{\omega}\left(\frac{t^2-2-t}{3}\right)\right) + \exp\left(-\frac{\Delta}{3\omega}\left(t^2-2\right)\right) \\
&\leq \exp\left(-\frac{\Delta}{3\omega}\left(t^2-3\right)\right) + \exp\left(-\frac{\Delta}{3\omega}\left(t^2-2\right)\right) \\
&\leq 2\exp\left(-\frac{\Delta}{3\omega}\left(t^2-3\right)\right).
\end{aligned}$$

□

Next, we present a theorem that extends Theorem 2 from Chung and Radcliffe (2011), which applies only to unweighted graphs, to symmetrically weighted graphs. Theorem A1 allows us to place tight bounds on the deviation of the realized induced adjacency matrix from its expected counterpart, drawing on the concentration bounds just derived.

Theorem A1. *Let $\mathcal{G}$ be an undirected random graph such that all edge formation probabilities are jointly independent. Denote by $\boldsymbol{A}$ the adjacency matrix and $\tilde{\boldsymbol{A}}$ the* *induced adjacency matrix by Definition 1. Let $\tilde{\boldsymbol{D}}$ be the diagonal degree matrix such that $\{\tilde{\boldsymbol{D}}\}_{ii} = \sum_j \tilde{A}_{ij}$ and $\{\tilde{\boldsymbol{D}}\}_{i\neq j} = 0$. Moreover, let $\tilde{\boldsymbol{L}} = \boldsymbol{I} - \tilde{\boldsymbol{D}}^{-1/2}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{D}}^{-1/2}$ denote the corresponding (normalized) Laplacian of $\mathcal{G}$, $w_{max} := \max_{i,j} w_{ij}$ be the largest weight, $w_{min} := \min_{i,j} w_{ij}$ the smallest weight, and $d_{min} := \min_i \sum_{j\in\mathcal{V}} w_{ij}p_{ij}$ the smallest expected degree. Let $\bar{\tilde{\boldsymbol{A}}}$, $\bar{\tilde{\boldsymbol{D}}}$, and $\bar{\tilde{\boldsymbol{L}}}$ denote the expected equivalents.*

Then for any $\psi > 0$, there exists a $k(\psi) > 0$ such that

$$\Pr\left(||\tilde{\boldsymbol{L}} - \bar{\tilde{\boldsymbol{L}}}|| \leq 4\sqrt{\frac{3w_{max}\ln(4n/\psi)}{d_{min}}}\right) \geq 1 - \psi$$

if Assumption 3 holds.

Proof. By the triangle inequality, for any matrix $\boldsymbol{Z}$,

$$||\tilde{\boldsymbol{L}} - \bar{\tilde{\boldsymbol{L}}}|| \leq ||\boldsymbol{Z} - \bar{\tilde{\boldsymbol{L}}}|| + ||\tilde{\boldsymbol{L}} - \boldsymbol{Z}||.$$

First, we bound $||\boldsymbol{Z} - \bar{\tilde{\boldsymbol{L}}}||$. In particular, let $\boldsymbol{Z} = \boldsymbol{I} - \bar{\tilde{\boldsymbol{D}}}^{-1/2}\tilde{\boldsymbol{A}}\bar{\tilde{\boldsymbol{D}}}^{-1/2}$. Then, since the degree matrices are diagonal, we have $\boldsymbol{Z} - \bar{\tilde{\boldsymbol{L}}} = \bar{\tilde{\boldsymbol{D}}}^{-1/2}(\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}})\bar{\tilde{\boldsymbol{D}}}^{-1/2}$. Denoting by $\boldsymbol{Y}_{ij}$ the matrix that is equal to 1 in the i,jth and j,ith positions and 0 elsewhere, we can use the symmetry of weights to write the i,jth entry of $\boldsymbol{Z} - \bar{\boldsymbol{L}}$ as

$$\boldsymbol{X}_{ij} = \bar{\tilde{\boldsymbol{D}}}^{-1/2}\left(w_{ij}(A_{ij} - p_{ij})\boldsymbol{Y}_{ij}\right)\bar{\tilde{\boldsymbol{D}}}^{-1/2} = \frac{w_{ij}(A_{ij} - p_{ij})}{\sqrt{\bar{d}_i\bar{d}_j}}\boldsymbol{Y}_{ij}.$$

where we denote $\bar{d}_i = \sum_{j\in\mathcal{V}} w_{ij}p_{ij}$ as the expected weighted degree of node i.

We know $\boldsymbol{Z} - \bar{\boldsymbol{L}} = \sum \boldsymbol{X}_{ij}$, so Lemma A1 applies. Since $\mathbb{E}(A_{ij}) = p_{ij}$, we have that $\mathbb{E}(\boldsymbol{X}_{ij}) = \boldsymbol{0}$, so that $v^2 = ||\sum \mathbb{E}(\boldsymbol{X}_{ij}^2)||$. Hence each $\boldsymbol{X}_{ij}$ is bounded above by $\|\boldsymbol{X}_{ij}\| \leq \frac{w_{max}}{d_{min}}$ and we have

$$\mathbb{E}(\boldsymbol{X}_{ij}^2) = \left[\bar{d}_i\bar{d}_j\right]^{-1} w_{ij}^2 p_{ij}(1 - p_{ij})(\boldsymbol{Y}_{ii} + \boldsymbol{Y}_{jj})$$

and then

$$
\begin{aligned}
v^2 &= \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i \bar{d}_j} p_{ij}(1-p_{ij}) \boldsymbol{Y}_{ii} \right\| \\
&= \max_i \left(\sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i \bar{d}_j} p_{ij}(1-p_{ij}) \right) \\
&\leq \max_i \left(\frac{w_{max}}{d_{min}} \sum_{j=1}^{n} \frac{w_{ij} p_{ij}}{\bar{d}_i} \right) \\
&= \frac{w_{max}}{d_{min}}
\end{aligned}
$$

Denote $a = \sqrt{\frac{3w_{max}\ln(4n/\psi)}{d_{min}}}$ and d_{min} so that $a < 1$. In particular, we must have $d_{min} > 3w_{max}(\ln(4) + \ln(n) - \ln(\psi))$, so that if $k \geq 3w_{max}(1 + \ln(4/\psi))$, $d_{min} \geq k w_{max} \ln(n)$ guarantees the result, which holds by Assumption 3. Then, by Lemma A1,

$$
\begin{aligned}
\Pr(\|\boldsymbol{Z} - \bar{\tilde{\boldsymbol{L}}}\| > a) &\leq 2n \exp\left(- \frac{\frac{3w_{max}\ln(4n/\psi)}{d_{min}}}{\frac{2w_{max}}{d_{min}}(1 + a/3)} \right) \\
&= 2n \exp\left(- \frac{3\ln(4n/\psi)}{2(1 + a/3)} \right) \\
&\leq 2n \exp\left(-\ln(4n/\psi) \right) \\
&= \frac{\psi}{2}.
\end{aligned}
$$

Now for the second term, we have

$$
\left\| \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I} \right\|_2 = \max_i \left| \sqrt{\frac{d_i}{\bar{d}_i}} - 1 \right| = \left\| \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I} \right\|_2 = \max_i \left| \sqrt{\frac{d_i/n}{\bar{d}_i/n}} - 1 \right|.
$$

Here, d_i is a sum of Bernoulli random variables that are bounded between 0 and w_{max}. Then, by Lemma A2, we have that for any $0 < t < 1$,

$$
\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq 2\exp\left(- \frac{\bar{d}_i(t^2 - 3)}{3w_{max}} \right).
$$

In particular, let $t = \sqrt{\frac{3w_{max}\ln(4n/(\psi \exp(d_{min}/w_{max})))}{d_{min}}} = a - \frac{1}{\sqrt{w_{max}}}$. We have $t < a < 1$ provided that $w_{max} > \frac{1}{a^2}$ or equivalently $w_{max} > \sqrt{\frac{d_{min}}{3\ln(4n/\psi)}}$, which holds by Assumption

3. For all i, we then obtain

$$\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq \frac{\psi}{2n}. \tag{A3}$$

Now, to bound

$$\left\| \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I} \right\|_2 = \max_i \left| \sqrt{\frac{d_i}{\bar{\bar{d}}_i}} - 1 \right|,$$

we can conclude by inequality (A3) that $\Pr\left(\left|d_i/\bar{d}_i - 1\right| > t\right) \leq \frac{\psi}{2}$ and hence with probability at least $1 - \frac{\psi}{2}$,

$$\left\| \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I} \right\|_2 < \sqrt{\frac{3w_{max} \ln\left(4n/\left(\psi \exp\left(d_{min}/w_{max}\right)\right)\right)}{d_{min}}}.$$

Then, for the second term, we have

$$\begin{aligned}
\|\tilde{\boldsymbol{L}} - \boldsymbol{Z}\| &= \|\boldsymbol{I} - \tilde{\boldsymbol{D}}^{-1/2} \tilde{\boldsymbol{A}} \tilde{\boldsymbol{D}}^{-1/2} - \boldsymbol{I} + \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{A}} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| \\
&= \|(\boldsymbol{I} - \tilde{\boldsymbol{L}}) \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} \tilde{\boldsymbol{D}}^{-1/2} \tilde{\boldsymbol{A}} \tilde{\boldsymbol{D}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| \\
&= \|(\boldsymbol{I} - \tilde{\boldsymbol{L}}) \bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} (\boldsymbol{I} - \boldsymbol{L}) \tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| \\
&= \|(\bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I})(\boldsymbol{I} - \tilde{\boldsymbol{L}}) \tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2} + (\boldsymbol{I} - \boldsymbol{L})(\boldsymbol{I} - \tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2})\| \\
&\leq \|\bar{\tilde{\boldsymbol{D}}}^{-1/2} \tilde{\boldsymbol{D}}^{1/2} - \boldsymbol{I}\| \|\tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| + \|\boldsymbol{I} - \tilde{\boldsymbol{D}}^{1/2} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| \\
&\leq t^2 + 2t
\end{aligned}$$

because $\|\boldsymbol{I} - \boldsymbol{L}\|_2 \leq 1$ from the fact that $\|\boldsymbol{L}\|_2 \leq 2$ (Chung and Graham, 1997). Finally, putting these together, we can conclude that

$$\begin{aligned}
||\tilde{\boldsymbol{L}} - \bar{\tilde{\boldsymbol{L}}}|| &\leq ||\boldsymbol{Z} - \bar{\tilde{\boldsymbol{L}}}|| + ||\tilde{\boldsymbol{L}} - \boldsymbol{Z}|| \\
&\leq 3a + a^2 + \frac{1}{w_{max}} - \frac{2 + 2a}{\sqrt{w_{max}}} \\
&\leq 4a + \frac{1}{\sqrt{w_{max}}} \left(\frac{1}{\sqrt{w_{max}}} - 2 \right) \\
&= 4a + \frac{1 - 2\sqrt{w_{max}}}{w_{max}} \\
&\leq 4a \\
&= 4\sqrt{\frac{3w_{max} \ln(4n/\psi)}{d_{min}}}
\end{aligned}$$

because $a < 1$ and $w_{max} \geq 1/4$. $\square$

With these preliminaries established, we are now ready to state the proofs of results presented in the main text.

Proof of Lemma 1. Let $\mathcal{G}_n$ be a sequence of random graphs over n vertices, and denote by $d_{min}^{(n)}$ the smallest expected weighted degree, i.e., $d_{min}^{(n)} := \min_i \sum_j w_{ij}^{(n)} p_{ij}^{(n)}$. Further, let $w_{max}^{(n)} = \max_{i,j} w_{ij}^{(n)}$ and $w_{min}^{(n)} = \min_{i,j} w_{ij}^{(n)}$ be the largest and smallest individual weights, satisfying $w_{min}^{(n)} \geq \omega$ for some $\omega > 0$ for all n. Then, if there exists a nondecreasing sequence of $k^{(n)} > 0$ such that $d_{min}^{(n)} \geq k^{(n)} \ln(n)$, the realized centrality vector centrality vector $\boldsymbol{c}(\tilde{\boldsymbol{A}}^{(n)})$ is with high probability close to the centrality of the average graph $\boldsymbol{c}(\bar{\tilde{\boldsymbol{A}}}^{(n)})$ for large n.

Under Assumption 3, we can apply Theorem A1 to conclude that, for any $\eta > 0$ and for all n, we have (omitting n notation on matrices for convenience),

$$\Pr\left(||\tilde{\boldsymbol{L}} - \bar{\tilde{\boldsymbol{L}}}|| \leq 4\sqrt{\frac{3w_{max}^{(n)} \ln(4n/\eta)}{d_{min}^{(n)}}} \right) \geq 1 - \eta$$

and by the assumption on the minimum degree's growth rate, $\lim_{n\to\infty} 4\sqrt{\frac{3w_{max}^{(n)} \ln(4n/\eta)}{d_{min}^{(n)}}} = 0$ regardless of the η chosen, so that under the 2-norm,

$$\tilde{\boldsymbol{L}} \underset{p}{\rightarrow} \bar{\tilde{\boldsymbol{L}}}.$$

For convenience call $\boldsymbol{B} = \boldsymbol{I} - \tilde{\boldsymbol{L}}$ and $\bar{\boldsymbol{B}}$ the expected equivalent. Clearly, we also have $\boldsymbol{B} \underset{p}{\rightarrow} \bar{\boldsymbol{B}}$, and can write $\boldsymbol{B} = \tilde{\boldsymbol{D}}^{-1/2} \tilde{\boldsymbol{A}} \tilde{\boldsymbol{D}}^{-1/2}$. Using properties of matrix norms (abusing notation in the second step slightly so that the maximum is over the norm of the matrices) and the above result, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| &= \limsup_{n\to\infty} \|\tilde{\boldsymbol{D}}^{-1/2} \boldsymbol{B} \tilde{\boldsymbol{D}}^{-1/2} - \bar{\tilde{\boldsymbol{D}}}^{-1/2} \bar{\boldsymbol{B}} \bar{\tilde{\boldsymbol{D}}}^{-1/2}\| \\
&\leq \limsup_{n\to\infty} \| \max\left\{\tilde{\boldsymbol{D}}^{-1/2}, \bar{\tilde{\boldsymbol{D}}}^{-1/2}\right\} (\boldsymbol{B} - \bar{\boldsymbol{B}}) \max\left\{\tilde{\boldsymbol{D}}^{-1/2}, \bar{\tilde{\boldsymbol{D}}}^{-1/2}\right\} \| \\
&\leq \limsup_{n\to\infty} \max\left\{\|\tilde{\boldsymbol{D}}^{-1/2}\|, \|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|\right\} \|\boldsymbol{B} - \bar{\boldsymbol{B}}\| \max\left\{\|\tilde{\boldsymbol{D}}^{-1/2}\|, \|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|\right\} \\
&\leq \limsup_{n\to\infty} \Psi \max\left\{\|\tilde{\boldsymbol{D}}^{-1/2}\|, \|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|\right\} \max\left\{\|\tilde{\boldsymbol{D}}^{-1/2}\|, \|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|\right\}.
\end{aligned}$$

Because Ψ can be chosen to be arbitrarily small, it is sufficient to establish that $\|\tilde{\boldsymbol{D}}^{-1/2}\|_2 \|\tilde{\boldsymbol{D}}^{-1/2}\|_2$ and $\|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|_2 \|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|_2$ are bounded by a constant almost surely. To see that they are, observe that

$$\begin{aligned}
\|\tilde{\boldsymbol{D}}^{-1/2}\|_2\|\tilde{\boldsymbol{D}}^{-1/2}\|_2 &= \sqrt{\frac{1}{\min_i \sum_j w_{ij}^{(n)} A_{ij}^{(n)}}}\sqrt{\frac{1}{\min_h \sum_j w_{hj}^{(n)} A_{hj}^{(n)}}} \\
&\leq \sqrt{\frac{1}{(w_{min}^{(n)})^2}} \\
&\leq \frac{1}{\omega}
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|_2\|\bar{\tilde{\boldsymbol{D}}}^{-1/2}\|_2 &= \sqrt{\frac{1}{\min_i \sum_j w_{ij}^{(n)} p_{ij}^{(n)}}}\sqrt{\frac{1}{\min_h \sum_j w_{hj}^{(n)} p_{hj}^{(n)}}} \\
&\leq \sqrt{\frac{1}{(d_{min}^{(n)})^2}} \\
&\leq 1.
\end{aligned}$$

Hence, given any $\Xi \in \{\omega^{-1}, 1\}$,

$$\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| \leq \limsup_{n\to\infty} \Psi\Xi = 0.$$

That is, the weighted adjacency matrix approaches its expected counterpart for large n.

We now wish to show that, for arbitrary $\psi > 0$,

$$\lim_{n\to\infty} \Pr(\|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| \geq \psi) = 0.$$

The key observation is that for any $\zeta > 0$, there exists sufficiently large n such that with probability approaching 1, $\|\tilde{\boldsymbol{A}}^h - \bar{\tilde{\boldsymbol{A}}}^h\| \leq \zeta$ for all $h \in \mathbb{Z}_+$. Because we have $\theta < 1$ by model assumptions, the formula for infinite geometric series can be applied,

$$\begin{aligned}
\limsup_{n\to\infty} \|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| &= \limsup_{n\to\infty} \left\|\sum_{h=0}^{\infty} \theta^h \left(\tilde{\boldsymbol{A}}^h - \bar{\tilde{\boldsymbol{A}}}^h\right)\right\| \\
&\leq \limsup_{n\to\infty} \sum_{h=0}^{\infty} |\theta^h| \left\|\left(\tilde{\boldsymbol{A}}^h - \bar{\tilde{\boldsymbol{A}}}^h\right)\right\| \\
&\leq \sum_{h=0}^{\infty} \zeta|\theta^h| \\
&= \frac{\zeta}{1-\theta}.
\end{aligned}$$

Moreover, because ζ was chosen arbitrarily, this implies that for any $\psi > 0$,

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{c}(\tilde{\boldsymbol{A}}^{(n)}) - \boldsymbol{c}(\bar{\tilde{\boldsymbol{A}}}^{(n)})\| > \psi) = 0.$$

Finally, note that $\theta > 0$ and Assumption 4 guarantee that the expected adjacency matrix has nonvanishing spectral gap (Mostagir and Siderius, 2021).

Denote by $P(i,j) = \frac{w_{ij}}{\sum_\ell w_{i\ell}}$ the *transition matrix* of a weighted graph. We additionally define a *circulation* of a directed graph $\mathcal{G}$ as a function $F : \mathcal{E}(\mathcal{G}) \to \mathbb{R}_{\geq 0}$ that assigns to each directed edge (i,j) a non-negative value such that

$$\sum_i F(i,j)\mathbb{1}(\{i,j\} \in \mathcal{E}) = \sum_h F(j,h)\mathbb{1}(\{j,h\} \in \mathcal{E}).$$

In particular, denote the circulation $F_{\boldsymbol{\lambda}}$ corresponding to stationary eigenvector $\boldsymbol{\lambda}$ having eigenvalue 1 as $F_{\boldsymbol{\lambda}}(i,j) = \boldsymbol{\lambda}(i)P(i,j)$.

Let S denote a subset of vertices. Then, we refer to the *out-boundary* of S as

$$F(\partial S) = \sum_{i\in S, j\notin S} F(i,j)$$

and define $F(S) = \sum_{j\in S}\sum_i F(i,j)\mathbb{1}(\{i,j\} \in \mathcal{E})$. Moreover, define the *Cheeger constant* (sometimes referred to as *conductance*) of $\mathcal{G}$ as

$$\varphi(\mathcal{G}) = \inf_S \frac{F_{\boldsymbol{\lambda}}(\partial S)}{\min\{F_{\boldsymbol{\lambda}}(S), F_{\boldsymbol{\lambda}}(S^c)\}}.$$

Now define a normalized (realized) adjacency matrix $\boldsymbol{T}$ with weights for each edge $\{i,j\}$ given by $\{\bar{\tilde{\boldsymbol{A}}}\bar{\tilde{\boldsymbol{D}}}^{-1}\}_{ij}$ with corresponding normalized Laplacian $\boldsymbol{L}_T = \boldsymbol{I} - \bar{\tilde{\boldsymbol{D}}}^{-1/2}\bar{\tilde{\boldsymbol{A}}}\bar{\tilde{\boldsymbol{D}}}^{-1/2}$. Following Mostagir and Siderius (2021), we apply Theorem 5.1 of Chung (2005) so that, with slight abuse of notation, we have a Cheeger inequality bounding $\varphi(\boldsymbol{T})$. Then $\varphi(\boldsymbol{T}) = \varphi(\bar{\tilde{\boldsymbol{A}}}\bar{\tilde{\boldsymbol{D}}}^{-1}) \geq \lambda_2^T/2$, where λ_2^T is the second eigenvalue of $\boldsymbol{T}$. Then, denote as $\upsilon = \lambda_1^T\lambda_2^T$, so that $\varphi(\bar{\tilde{\boldsymbol{A}}}\bar{\tilde{\boldsymbol{D}}}^{-1}) \geq (1-\upsilon)/2 > 0$.

Next, consider a network $\bar{\boldsymbol{T}}^* = \boldsymbol{T}\bar{\tilde{\boldsymbol{D}}}(\bar{\tilde{\boldsymbol{D}}}^*)^{-1}$, which is symmetric by construction. By Theorem A1 and the fact that $\boldsymbol{T}^* = \boldsymbol{T}\bar{\tilde{\boldsymbol{D}}}(\tilde{\boldsymbol{D}}^*)^{-1}$ and the corresponding $\boldsymbol{I} - \boldsymbol{L}_{T^*}$ have the same eigenvalues,

$$\|\lambda_t^{T^*} - \lambda_t^{\bar{T}^*}\| \leq 4\sqrt{\frac{3w_{max}^{(n)}\ln(4n/\psi)}{d_{min}^{(n)}}}$$

for $t = 1, 2$. Hence, with probability approaching 1, $\boldsymbol{T}^*$ does not have a vanishing spectral gap and is connected by the Cheeger inequality. Because $\boldsymbol{T}^*$ is connected if and only if $\boldsymbol{T}$ is, and hence if $\tilde{\boldsymbol{A}}$ is, the network is connected with high probability. Therefore, the centrality measure is well-defined. $\square$

Having determined that the realized centrality can be well-approximated by expected centrality, the following proof establishes our approach to calculating expected centrality for large networks.

Proof of Proposition 1. Take any graph $\mathcal{G}^{(n)}$ of size n. By the result established in Lemma 1, it is sufficient to consider centrality on the average network. We suppress notation of n for convenience and denote $\rho := w_H p_H$ and $\delta := \frac{1}{\rho} w_L p_L$ for some $0 < \delta < 1$. Additionally, we denote s_i the size of group i such that $s_1 = s$ and $s_2 = 1 - s$ and $n_i = ns_i$ the corresponding number of voters in group i.

We can then write the matrix $\boldsymbol{I} - 2\theta\bar{\tilde{\boldsymbol{A}}}$ as a 2×2 block matrix with blocks

$$
\begin{aligned}
\bar{\tilde{\boldsymbol{A}}}_{11} &= \boldsymbol{I} - \frac{2\theta}{n_1 + \delta n_2}\left(\boldsymbol{1}_{s_1 n \times s_1 n} - \boldsymbol{I}\right) \\
\bar{\tilde{\boldsymbol{A}}}_{12} &= -\frac{2\theta\delta}{n_1 + \delta n_2}\boldsymbol{1}_{s_1 n \times s_2 n} \\
\bar{\tilde{\boldsymbol{A}}}_{21} &= -\frac{2\theta\delta}{n_2 + \delta n_1}\boldsymbol{1}_{s_2 n \times s_1 n} \\
\bar{\tilde{\boldsymbol{A}}}_{22} &= \boldsymbol{I} - \frac{2\theta}{n_2 + \delta n_1}\left(\boldsymbol{1}_{s_2 n \times s_2 n} - \boldsymbol{I}\right).
\end{aligned}
$$

To apply the formula for block inversion, we first want to identify $\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}$. We conjecture

$$
P = \bar{\tilde{\boldsymbol{A}}}_{11}^{-1} = \begin{bmatrix} a_1 & q_1 & \cdots & q_1 \\ q_1 & a_1 & \cdots & q_1 \\ \vdots & \cdots & \ddots & \vdots \\ q_1 & \cdots & \cdots & a_1 \end{bmatrix}.
$$

Then, we have that

$$
\begin{bmatrix} 1 & -(n_1 - 1)\frac{2\theta\xi_1}{n_1 + \delta n_2} \\ -\frac{2\theta\xi_1}{n_1 + \delta n_2} & 1 - (n_1 - 2)\frac{2\theta\xi_1}{n_1 + \delta n_2} \end{bmatrix} \begin{bmatrix} a_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

which indeed has a unique solution. The inverse of the bottom-right block is identical, swapping group indices. Hence, we can construct the centrality vector by

(A4)

$$
\boldsymbol{I} - 2\theta\bar{\tilde{\boldsymbol{A}}} = \begin{bmatrix} \left(\bar{\tilde{\boldsymbol{A}}}_{11} - \bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1}\bar{\tilde{\boldsymbol{A}}}_{21}\right)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\bar{\tilde{\boldsymbol{A}}}_{22} - \bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}\bar{\tilde{\boldsymbol{A}}}_{12}\right)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1} \\ -\bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1} & \boldsymbol{I} \end{bmatrix}
$$

The main-diagonal blocks in the first matrix on the right-hand side of equation (A4) have the same structure, with a single value on the main diagonal and another value on the off-diagonal. This has a similar structure to $\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}$, and the inverse can thus be calculated analogously by solving for main and off-diagonal elements a'_i and q'_i.[6]

Substituting these values into equation (A4), we then have that

$$\hat{c}_i^{(n)}(\cdot) = (1 + n_{-i})a'_i + (1 + n_{-i})(n_i - 1)q'_i.$$

This expression has a substantive interpretation: a'_i is the weighted average of the number of paths back to a voter in group i through the network, while q'_i is the weighted average of the number of paths to someone else in i's group through the network. Hence, i's centrality does not depend at all on their paths to the other group. This is because each voter is connected to all others on the expected network, so that all paths within a voter's group corresponds to an equivalent cross-party path.

Since n is large by assumption, this expression is asymptotically equivalent to its leading term, which yields the desired expressions. □

The final lemma allows us to conclude that individual vote probabilities are approximately asymptotically independent of other voters' types, allowing us to treat the case of heterogeneous information similarly to the baseline.

Proof of Lemma 2. Consider any two vectors $\boldsymbol{\theta}^{(n)}$ and $\boldsymbol{\theta}'^{(n)}$ such that $\theta_i^{(n)} \neq \theta_i'^{(n)}$ for at least one $i \in \mathcal{V}$, with corresponding equilibrium vote probabilities $\boldsymbol{\phi}^{(n)}$ and $\boldsymbol{\phi}'^{(n)}$. Then by Assumption 5, we have that

$$\bar{\theta}^{(n)} - \mu^{min}(\bar{\theta}^{(n)} - \underline{\theta}^{(n)}) = \frac{\nu}{\chi n}.$$

It then follows that

$$\bar{\theta}^{(n)} - \underline{\theta}^{(n)} = \frac{\bar{\theta}^{(n)}\chi n - \nu}{\mu^{min}\chi n} = o(1)$$

and hence $\lim_{n\to\infty} |\bar{\theta}^{(n)} - \underline{\theta}^{(n)}| = 0$, so that also $\left|\theta^{(n)} - \theta'^{(n)}\right| \underset{n}{\rightarrow} 0$.

Now, we can write any i's probability of voting for candidate 1 under information profile θ (suppressing superscripts for readability) as

$$\phi_i = \frac{1}{2} + \theta_i \left((-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \sum_j w_{ij}\left(2\phi_j - 1\right) + \gamma \sum_m b_{m2} - b_{m1} \right)$$

[6] A unique solution exists, but we suppress the expression as it is highly complex and uninformative. Refer to the replication materials for more information.

and similarly for $\boldsymbol{\phi}'$ given θ'. Then we can write, for any $i \in \mathcal{V}$ such that $\theta_i \neq \theta_i'$,

$$
\begin{aligned}
|\phi_i - \phi_i'| \leq \big(\theta_i - \theta_i'\big) & \left((-1)^{x_i - 1} - \sum_j w_{ij} \right) \\
& + \theta_i \left(u(b_{i1}) - u(b_{i2}) + \sum_j 2 w_{ij} \phi_j + \gamma \sum_m b_m \right) \\
& - \theta_i' \left(u(b_{i1}) - u(b_{i2}) + \sum_j 2 w_{ij} \phi_j' + \gamma \sum_m b_m \right)
\end{aligned}
$$

Considering the first term, we have from above that $(\theta_i - \theta_i') = O(1/n)$, so that the term converges to 0 if $(-1)^{x_i-1} - \sum_j w_{ij} = o(n)$. Given the assumption of asymptotically constant weights, this is satisfied such that $\lim_{n\to\infty}(\theta_i - \theta_i')\big((-1)^{x_i-1} - \sum_j w_{ij}\big) = 0$.

For the second term, we again have that $\theta = O(1/n)$, so that we require

$$
\left(u(b_{i1}) - u(b_{i2}) + \sum_j 2 w_{ij} \phi_j + \gamma \sum_j b_{j1} - b_{j2} \right) = o(n)
$$

Since $\phi_j \leq 1$ for all j, it follows from the same argument as above that $\sum_j 2 w_{ij} \phi_j = o(n)$. Since transfers are taken as given, we can again conclude that

$$
\lim_{n\to\infty} \theta \left(u(b_{i1}) - u(b_{i2}) + \sum_j 2 w_{ij} \phi_j + \gamma \sum_j b_{j1} - b_{j2} \right) = 0.
$$

An identical argument establishes the same is true for the θ' term. Hence, we can conclude that $\lim_{n\to\infty} |\phi_i - \phi_i'| = 0$, completing the proof. □

Given the above lemma, the proof of the following proposition is largely analogous to the proof of Proposition 1 for the baseline model. However, we also need to establish that Lemma 1 continues to hold under heterogeneous information.

Proof of Proposition 2. It follows from Theorem A1 and Lemma 1 that we can again consider the expected network only. To see this, using an argument analogous to Part 2 of the proof of Theorem 1 in Mostagir and Siderius (2021), note that we can define $\boldsymbol{A}_\theta = \boldsymbol{\Theta}\boldsymbol{A}$, and analogously for $\tilde{\boldsymbol{A}}_\theta$, $\bar{\boldsymbol{A}}_\theta$, and $\tilde{\bar{\boldsymbol{A}}}_\theta$.

Then, for any $\zeta > 0$, we can write

$$\begin{aligned}
\limsup_{n\to\infty} \|(\boldsymbol{I} - \tilde{\boldsymbol{A}}_\theta)^{-1} - (\boldsymbol{I} - \bar{\tilde{\boldsymbol{A}}}_\theta)^{-1}\| &= \limsup_{n\to\infty} \left\| \sum_{h=0}^{\infty} \left(\tilde{\boldsymbol{A}}_\theta^h - \bar{\tilde{\boldsymbol{A}}}_\theta^h \right) \right\| \\
&\le \limsup_{n\to\infty} \sum_{h=0}^{\infty} \|\boldsymbol{\Theta}^h\| \left\| \left(\tilde{\boldsymbol{A}}^h - \bar{\tilde{\boldsymbol{A}}}^h \right) \right\| \\
&\le \limsup_{n\to\infty} \sum_{h=0}^{\infty} \zeta \left(\sup_i \theta_i^{(n)} \right)^h \\
&= \limsup_{n\to\infty} \frac{\zeta}{1 - \sup_i \theta_i^{(n)}},
\end{aligned}$$

where the second line follows from the proof of Lemma 1. Since ζ was chosen arbitrarily and $\limsup_{n\to\infty} \theta_i^{(n)} = 0$, this again implies that for any $\psi > 0$,

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}}) - \boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})\| > \psi) = 0.$$

The remainder of the proof proceeds analogously to that for Proposition 1, where we replace θ with $\boldsymbol{\Theta}$ accordingly. □

REFERENCES

Alatas, Vivi, Abhijit Banerjee, Rema Hanna, Benjamin A Olken, Ririn Purnamasari, and Matthew Wai-Poi. 2019. "Does Elite Capture Matter? Local Elites and Targeted Welfare Programs in Indonesia." *AEA Papers and Proceedings*, 109: 334–339.

Alesina, Alberto, Arnaud Devleeschauwer, William Easterly, Sergio Kurlat, and Romain Wacziarg. 2003. "Fractionalization." *Journal of Economic Growth*, 8(2): 155–194.

Alesina, Alberto, Reza Baqir, and William Easterly. 1999. "Public goods and ethnic divisions." *The Quarterly Journal of Economics*, 114(4): 1243–1284.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou. 2006. "Who's who in networks. Wanted: The key player." *Econometrica*, 74(5): 1403–1417.

Banks, Jeffrey S. 2000. "Buying supermajorities in finite legislatures." *American Political Science Review*, 94(3): 677–681.

Battaglini, Marco, and Eleonora Patacchini. 2018. "Influencing connected legislators." *Journal of Political Economy*, 126(6): 2277–2322.

Battaglini, Marco, Valerio Leone Sciabolazza, and Eleonora Patacchini. 2020. "Effectiveness of connected legislators." *American Journal of Political Science*, 64(4): 739–756.

Caeyers, Bet, and Stefan Dercon. 2012. "Political connections and social networks in targeted transfer programs: Evidence from rural Ethiopia." *Economic Development and Cultural Change*, 60(4): 639–675.

Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou. 2009. "Peer effects and social networks in education." *The Review of Economic Studies*, 76(4): 1239–1267.

Canen, Nathan, Jacob Schwartz, and Kyungchul Song. 2020. "Estimating local interactions among many agents who observe their neighbors." *Quantitative Economics*, 11(3): 917–956.

Canen, Nathan, Matthew O Jackson, and Francesco Trebbi. 2023. "Social interactions and legislative activity." *Journal of the European Economic Association*, 21(3): 1072–1118.

Chen, Ying-Ju, Yves Zenou, and Junjie Zhou. 2022. "The impact of network topology and market structure on pricing." *Journal of Economic Theory*, 204: 105491.

Chung, Fan. 2005. "Laplacians and the Cheeger inequality for directed graphs." *Annals of Combinatorics*, 9: 1–19.

Chung, Fan, and Mary Radcliffe. 2011. "On the spectra of general random graphs." *The Electronic Journal of Combinatorics*, P215–P215.

Chung, Fan RK, and Fan Chung Graham. 1997. *Spectral graph theory.* American Mathematical Soc.

Cruz, Cesi. 2019. "Social networks and the targeting of vote buying." *Comparative Political Studies*, 52(3): 382–411.

Cruz, Cesi, Julien Labonne, and Pablo Querubin. 2020. "Social network structures and the politics of public goods provision: evidence from the Philippines." *American Political Science Review*, 114(2): 486–501.

Dasaratha, Krishna. 2020. "Distributions of centrality on networks." *Games and Economic Behavior*, 122: 1–27.

Dekel, Eddie, Matthew O Jackson, and Asher Wolinsky. 2008. "Vote buying: General elections." *Journal of Political Economy*, 116(2): 351–380.

Downs, Anthony. 1957. "An economic theory of political action in a democracy." *Journal of political economy*, 65(2): 135–150.

Easterly, William, and Ross Levine. 1997. "Africa's growth tragedy: policies and ethnic divisions." *The quarterly journal of economics*, 1203–1250.

Elliott, Matthew, and Benjamin Golub. 2019. "A network approach to public goods." *Journal of Political Economy*, 127(2): 730–776.

Enikolopov, Ruben. 2014. "Politicians, bureaucrats and targeted redistribution." *Journal of Public Economics*, 120: 74–83.

Fafchamps, Marcel, and Julien Labonne. 2017. "Do politicians' relatives get better jobs? evidence from municipal elections." *The Journal of Law, Economics, and Organization*, 33(2): 268–300.

Gans-Morse, Jordan, Sebastian Mazzuca, and Simeon Nichter. 2014. "Varieties of clientelism: Machine politics during elections." *American Journal of Political Science*, 58(2): 415–432.

Ghasemian, Amir, Homa Hosseinmardi, and Aaron Clauset. 2019. "Evaluating overfit and underfit in models of network community structure." *IEEE Transactions on Knowledge and Data Engineering*, 32(9): 1722–1735.

Golub, Benjamin, and Matthew O Jackson. 2012. "How homophily affects the speed of learning and best-response dynamics." *The Quarterly Journal of Economics*, 127(3): 1287–1338.

Groseclose, Tim, and James M Snyder. 1996. "Buying supermajorities." *American Political Science Review*, 90(2): 303–315.

Lee, Lung-Fei, Xiaodong Liu, Eleonora Patacchini, and Yves Zenou. 2021. "Who is the key player? A network analysis of juvenile delinquency." *Journal of Business & Economic Statistics*, 39(3): 849–857.

Lizzeri, Alessandro, and Nicola Persico. 2001. "The provision of public goods under alternative electoral incentives." *American Economic Review*, 91(1): 225–239.

Mostagir, Mohamed, and James Siderius. 2021. "Centrality in stochastic networks." *Working paper.*

Myerson, Roger B. 1993. "Incentives to cultivate favored minorities under alternative electoral systems." *American Political Science Review*, 87(4): 856–869.

Sadler, Evan. 2023. "Influence campaigns." *American Economic Journal: Microeconomics*, 15(3): 271–304.

Schaeffer, Merlin. 2013. "Ethnic diversity, public goods provision and social cohesion: Lessons from an inconclusive literature." WZB Discussion Paper.

Singh, Prerna, and Matthias Vom Hau. 2016. "Ethnicity in time: Politics, history, and the relationship between ethnic diversity and public goods provision." *Comparative Political Studies*, 49(10): 1303–1340.

Vaca-Ramírez, Felipe, and Tiago P Peixoto. 2022. "Systematic assessment of the quality of fit of the stochastic block model for empirical networks." *Physical Review E*, 105(5): 054311.

Wang, Ruxian, and Zizhuo Wang. 2017. "Consumer choice models with endogenous network effects." *Management Science*, 63(11): 3944–3960.

Yildiz, Ercan, Asuman Ozdaglar, Daron Acemoglu, Amin Saberi, and Anna Scaglione. 2013. "Binary opinion dynamics with stubborn agents." *ACM Transactions on Economics and Computation (TEAC)*, 1(4): 1–30.